Global Analysis I, lecture 1

10. 10. 2017.

Practicalities

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Course webpage (to be linked in BASIS): www. math. uni-bonn. de / people / mesland / F4B1 – V3B3. html

Assignments posted every Wednesday night, assignments are due on the next Wednesday lecture

Literature: John M. Lee: Intro to smooth manifolds (first couple of lectures)
— " —: Riemannian manifolds (rest of the lecture)

To take the exam, a 50% score is required on the assignments.

Tutorial: Mo 10 – 12 NO.007 EA60 (conflicts with the Topology I lecture)
Saskia Roos

"If you want an electronic copy of these books, please email me, although I think you know your way around the internet. We live in a beautiful world, don't we?"

The new time slot for the Tuesday lecture will be announced tonight.

— ∘ —

Smooth manifolds (review)

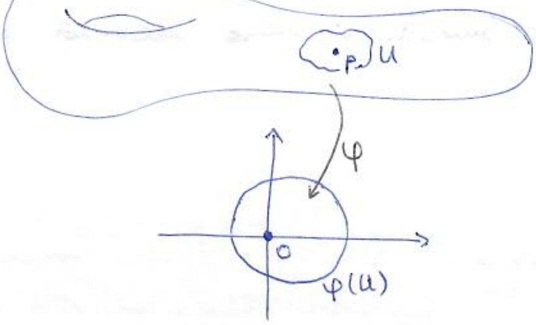
Def. Topological manifold: second-countable top. space M that is Hausdorff and locally Euclidean.

A top. space M is loc. Eucl. if $\forall p \in M$ $\exists U \subset M$ open and $V \subset \mathbb{R}^n$ (for some n) open and $\exists \varphi: U \to V$ homeomorphism. (We often assume $\varphi(p) = 0$. Quite often $V = B(0,1)$.)

Lemma. (PO) (Consequence of 2nd countability) Every open cover of M has a countable subcover.

Def. A chart around $p \in M$ is a pair (U, φ) with $p \in U \subset M$ open, $\varphi: U \to \varphi(U) \subset \mathbb{R}^n$ a homeomorphism. The set U is sometimes called a coordinate domain. and $\varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix} \in \mathbb{R}^n$, φ_i are called local coordinates.

If $\varphi(U) = B(0, R)$ for some R and $\varphi(p) = 0$, then we say that the coordinates φ_i are centered at p.



Example. $S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$, $\|x\| = \sum_{i=1}^{n+1} |x_i|^2$

S^n is 2nd-countable, Hausdorf, because it is closed in $\mathbb{R}^{n+1}$.

For the local Euclideanness:

$$U_i^{\pm} := \{ (x_1, \dots, x_{n+1}) \in S^n \mid \pm x_i > 0 \}$$

$\Rightarrow$ $U_i^+ = pr_i^{-1}(0, \infty)$ and $U_i^- = pr_i^{-1}(-\infty, 0)$

$\Rightarrow$ $U_i^{\pm}$ are open

this coordinate is omitted

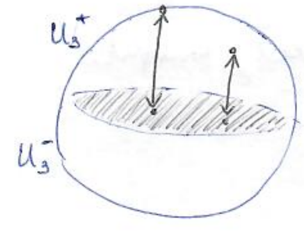
$$\varphi_i^{\pm}(x_1, \dots, x_{n+1}) := (x_1, \dots, x_{i-1}, \widehat{x_i}, x_{i+1}, \dots, x_{n+1})$$

$$\varphi_i^{\pm} : U_i^{\pm} \longrightarrow B^n = \{ (x_1, \dots, x_n) \mid \|x\| < 1 \} \subset \mathbb{R}^n$$

In fact, $\varphi_i^{\pm}$ is a homeomorphism because the inverse is

$$(\varphi_i^{\pm})^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, \pm\sqrt{x_1^2 + \dots + x_n^2}, x_i, \dots, x_n) \in \mathbb{R}^{n+1}.$$

The inverse is well defined and continuous as well.



We have not yet said anything about differentiability.

Let (U_i, φ_i) and (U_j, φ_j) be two coordinate charts.

We say that these charts are <u>C^k-compatible</u> if

$$\varphi_j \circ \varphi_i^{-1} : \underbrace{\varphi_i(U_i \cap U_j)}_{\subset \mathbb{R}^n} \longrightarrow \underbrace{\varphi_j(U_i \cap U_j)}_{\subset \mathbb{R}^n}$$ is a C^k-map, with a C^k inverse.

(Of course, $U_i \cap U_j$ might be $\emptyset$.)

We say that they are smoothly compatible if the above map is C^∞, and its inverse is C^∞ as well, so in other words, $\varphi_j \circ \varphi_i^{-1}$ is a diffeomorphism.

Def. An atlas for a manifold M is a collection of charts $\{(U_i, \varphi_i)\}$ such that $\{U_i\}_{i \in I}$ is a cover of M.

The atlas is C^k or smooth if all of its charts are C^k / C^∞-compatible.

A C^k-atlas is maximal if it is not contained in any other C^k-atlas. (same for smooth)

A C^k-structure on M is a choice of maximal C^k-atlas. (same for smooth)

Note. A smooth structure need not exist nor is it unique.

Global Analysis I, lecture 1

Def. A fuction $f: M \to \mathbb{R}$ on a smooth manifold M is smooth if for every smooth chart (U_i, φ_i) the map $f \circ \varphi_i^{-1} : \underbrace{\varphi_i(U_i)}_{\subset \mathbb{R}^n} \to \mathbb{R}$ is a smooth map

$C^\infty(M) := \{ f : M \to \mathbb{R} \mid f \text{ smooth} \} \subset C(M) = \{ f : M \to \mathbb{R} \mid f \text{ continuous} \}$

$C^\infty(M)$ is an $\mathbb{R}$-algebra (that is, $\forall f, g \in C^\infty(M)$, $\mu, \lambda \in \mathbb{R}$: $f+g$, $f \cdot g$, $\lambda f + \mu g \in C^\infty(M)$ as well).

Example. Any open set $U \subset \mathbb{R}^n$ is a smooth manifold: the chart (U, id) is enough to see this.

Lemma. 1) Every smooth atlas is contained in a unique maximal smooth atlas.
2) Two smooth atlases determine the same maximal smooth atlas iff their union is a smooth atlas.

Example. S^n: the charts $U_i^\pm$ are smoothly compatible. Assume $i<j$, and let us consider on $\varphi_i(U_i^+ \cap U_j^+)$ the map $\varphi_j^+ \circ (\varphi_i^+)^{-1}$:

$$\varphi_j^+ \left((\varphi_i^+)^{-1} (u_1, \dots, u_n) \right) = (u_1, \dots, u_{i-1}, \sqrt{1-|u|^2}, u_i, \dots, \widehat{u_j}, \dots, u_n).$$

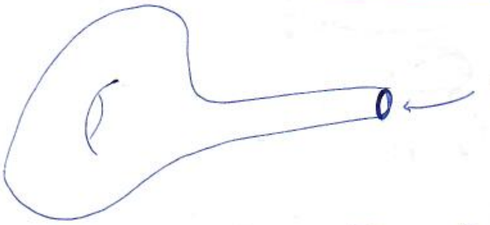
Similarly for $i>j$ and for $-$ instead of $+$.

For $i=j$: $\varphi_j^+ \circ (\varphi_i^+)^{-1} = id_{B^n}$.

Thus $\{ (U_i^\pm, \varphi_i^\pm) \}_{i=1}^{n+1}$ is a smooth atlas.

This atlas describes S^n locally as the graph of the fuction $u \mapsto \pm \sqrt{1-|u|^2}$

Manifolds with boundary



the points on the boundary only have neighbourhoods like a half-space.

Write $\overline{H}^n = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geqq 0 \}$ for the closed upper half space in $\mathbb{R}^n$. (with the relative topology)

$\partial \overline{H}^n = \{ (x_1, \dots, x_n) \mid x_n = 0 \}$

Def. A top. manifold with boundary is a 2nd-countable, Hausdorff space M that is locally homeomorphic to an open subset of $\overline{H}^n$.

A pair (U, φ), $\varphi : U \to \varphi(U) \subset \overline{H}^n$, $U \subset M$ open is a generalized chart if φ is a homeomorphism.

Def. A **boundary point** of M is a point p for which exists a gen. chart (U, φ) with $\varphi(p) = (x_1, \dots, x_{n-1}, 0) \in \partial \bar{H}^n$.

The **boundary of M** is the collection of such boundary points:

$\partial M = \{ p \in M \mid p \text{ is a boundary point} \}$

$p \in M$ is an **interior point of M** if $\exists$ a gen chart s.t. $\varphi(p) = (x_1, \dots, x_n)$, $x_n > 0$.

$\text{int } M = \{ p \in M \mid p \text{ is an interior point} \}$

Question. $\text{int } M \cap \partial M = \emptyset$? (yes, if M is smooth — to be discussed later)

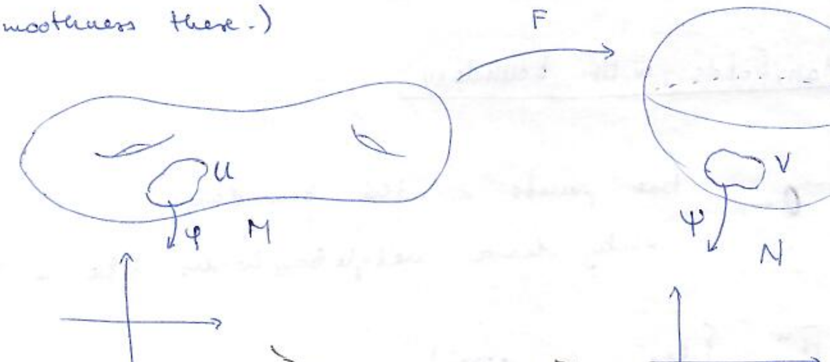
Warning: ∂M is not the topological boundary of M (which is $\bar{U} \setminus U^\circ$)

Def. A **smooth structure** on a top. manifold with boundary M is a choice of maximal smooth atlas, where two gen. charts (U_i, φ_i), (U_j, φ_j) are smoothly compatible if $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ admits a smooth extension to some open set $W \subset \mathbb{R}^n$ with $\varphi_i(U_i \cap U_j) \subset W$.

Smooth maps and diffeomorphisms

Def. Let M, N be smooth manifolds (without boundary), and $F: M \to N$ a continuous map.*
Then F is **smooth** if $\forall$ smooth chart (U, φ) on M and (V, ψ) s. chart on N the map $\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$ is smooth.

(We basically pull the whole thing back to $\mathbb{R}^n$ using charts and check the smoothness there.)



* Acc. to Meslandi, continuity should somehow follow.

Composition of smooth maps:

$F: M \to N, \quad G: N \to P \implies G \circ F : M \to P$ is smooth

Example. $i: S^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth map. (inclusion)

$i \circ (\varphi_i^{\pm})^{-1} : \varphi_i(U_i^{\pm}) \to \mathbb{R}^{n+1}$

$(u_1, \dots, u_n) \mapsto (u_1, \dots, \pm\sqrt{1 - |u|^2}, \dots, u_n)$ is smooth since $|u|^2 < 1$.

Global Analysis 1-2

Def. A diffeomorphism is a smooth bijection with smooth inverse.

Example. The map $F: B^n \longrightarrow \mathbb{R}^n$ is a diffeomorphism.
$$x \longmapsto \frac{1}{1-\|x\|^2}$$

Bump functions and partitions of unity

11.10.2017

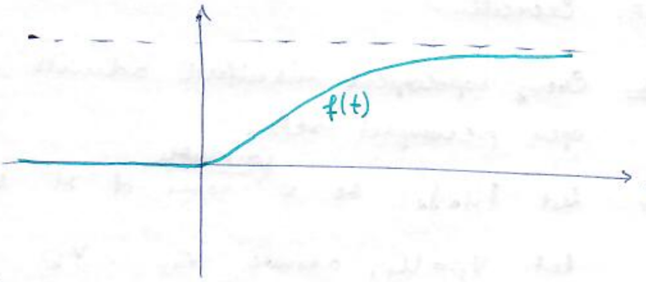
Def. $f: M \to \mathbb{R}^n$, supp $f := \overline{\{p \in M \mid f(p) \neq 0\}}$ support.

If supp $f \subseteq U$, we say that f is supported in U,

if supp f is compact, we say that f is compactly supported.

Lemma. $f: \mathbb{R} \to \mathbb{R}$, $f(t) = \begin{cases} e^{-1/t} & t>0 \\ 0 & t \leq 0 \end{cases}$ is smooth.

PROOF: Smoothness is trivial by composition on $\mathbb{R} \setminus \{0\}$.



Suffices to show that the k^{th} derivative exists for all $k \in \mathbb{N}$; then the $(k-1)^{th}$ derivative is continuous.

f is cont. at 0: $\lim\limits_{t \searrow 0} e^{-1/t} = 0$.

We show by induction that $f^{(k)}(t) = p_k(t) \cdot \frac{e^{-1/t}}{t^{2k}}$ $(t>0)$

where $p_k(t) \in \mathbb{R}[t]$, $\deg p_k \leq k$. True for $k = 0$.

Product rule: $f^{(k+1)}(t) = p'_k(t)\frac{e^{-1/t}}{t^{2k}} + p_k \frac{t^{-2}e^{-1/t}}{t^{2k}} - 2k\, p_k(t) \frac{e^{-1/t}}{t^{2k+1}} =$
$$= \left(t^2 p_k'(t) + p_k(t) - 2k t\, p_k(t)\right) \cdot \frac{e^{-1/t}}{t^{2(k+1)}}$$

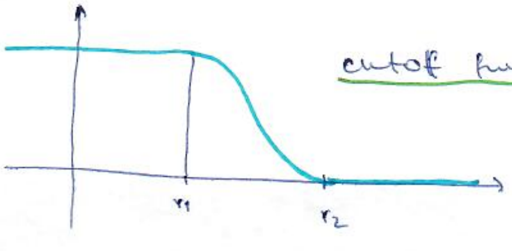
Induction: $f^{(k)}(0) = 0 \quad \forall k \geq 0$. True for $k=0$.

→ show that $f^{(k+1)}(0)$ exists and is zero: both one-sided derivatives exist and are zero. This is trivial for the left.

$$\lim_{t \searrow 0} \frac{p_k(t) \cdot \frac{e^{-1/t}}{t^{2k}} - 0}{t} = \lim_{t \searrow 0} p_k(t) \cdot \frac{e^{-1/t}}{t^{2k+1}} = p_k(0) \cdot \lim_{t \searrow 0} \frac{e^{-1/t}}{t^{2k+1}} = 0.$$ □

Lemma. $\forall r_1 < r_2 : \exists h: \mathbb{R} \to \mathbb{R}$: $h(t) = 1 \quad \forall t \leq r_1$,
$h(t) = 0 \quad \forall t \geq r_2$,
$0 \leq h(t) \leq 1 \quad \forall r_1 < t < r_2$, h smooth.

PROOF: $h(t) = \dfrac{f(r_2 - t)}{f(r_2-t) + f(t-r_1)}$



cutoff function □

Lemma. $\forall r_1 < r_2 \quad \exists H: \mathbb{R}^n \to \mathbb{R}: \quad H\big|_{\overline{B}_{r_1}(0)} \equiv 1, \quad H\big|_{\mathbb{R}^n \setminus B_{r_2}(0)} \equiv 0,$

$0 < H(x) < 1 \quad \forall x \in B_{r_2}(0) \setminus \overline{B}_{r_1}(0), \quad H$ smooth.

PROOF: $H(x) = h(|x|)$. $\to$ H is smooth on $\mathbb{R}^n \setminus \{0\}$, and constant around 0 $\Rightarrow$ smooth on $\mathbb{R}^n$. □

Def. M top. space, $\{U_\alpha\}_{\alpha \in A}$ a collection. This collection is locally finite if $\forall p \in M$ $\exists V$ nbh. s.t. $V \cap U_\alpha \neq \emptyset$ for at most finitely many $\alpha \in A$.

Lemma. $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ open precompact sets.
Then $\mathcal{U}$ is locally finite $\iff$ $\forall \alpha$ there are at most finitely many $\beta \in A$ s.t. $U_\alpha \cap U_\beta \neq \emptyset$.

PROOF: Exercise. □

Lemma. Every topological manifold admits a locally finite cover by open precompact sets.

PROOF: Let $\{U_i\}_i$ be a countable cover of M by precompact opens; e.g. U_α are balls.

Let $V_1 := U_1$, assume $V_1, \dots, V_k$ are defined s.t. $U_j \subset V_j$ and $\overline{V_{j-1}} \subset V_j$.

$\overline{V_k}$ cpt. $\Rightarrow$ $\overline{V_k} \subset U_1 \cup \dots \cup U_{m_k}$ (We may assume $m_k > k$).

$V_{k+1} := U_1 \cup \dots U_{m_k}$. V_{k+1} is cpt. since $\overline{V}_{k+1} = \overline{U}_1 \cup \dots \cup \overline{U_{m_k}}$

Since $U_j \subset V_j$, $\{V_j\}_j$ covers M.

Let $W_j := V_j \setminus \overline{V}_{j-2}$. W_j is precompact, since $\overline{W_j}$ is compact because it is a closed subset of $\overline{V_j}$ which is cpt.

$\{W_j\}_j$ is a locally finite cover. □

Def. $\mathcal{V} = \{V_\beta\}_\beta$ open cover is a refinement of the open cover $\mathcal{U} = \{U_\alpha\}_\alpha$ if $\forall V_\beta \ \exists U_\alpha: V_\beta \subset U_\alpha$.

Def. M top. space is paracompact if every open cover admits a loc. finite refinement.

Def. M manifold, $\mathcal{W} = \{W_i\}_{i \in I}$ an open cover. Then $\mathcal{W}$ is regular if

1) $\mathcal{W}$ is countable and loc. finite,
2) $\forall i \ \exists \psi_i: W_i \to B(0,3) \subset \mathbb{R}^n$ diffeomorphism,
3) $U_i := \psi_i^{-1}(B(0,1))$ covers M.

Global Analysis 1, lecture 2

Proposition. Let M be a smooth manifold. Then every open cover admits a regular refinement.

PROOF: Let X be a collection of open sets which cover M,
$\{V_j\}$ a countable loc. fin cover of M, $\forall$ V_j precompact open.

$\forall p \in M$: let (W_p, ψ_p) be a chart centered at p s.t. $\psi_p(W_p) = B_3(0)$, W_p is contained in an open set in X and if $p \in V_j$ then $W_p \subset V_j$ (this is possible because $\{V_j\}_j$ is loc. finite).

$U_p := \psi_p^{-1}(B_1(0))$.

$\forall k$: $\{U_p \mid p \in \overline{V_k}\}$ is an open cover of $\overline{V_k}$.

$\overline{V_k}$ cpt. $\Rightarrow$ $U_{k,1}, \dots, U_{k,m(k)}$ finite subcover obtained from $(W_{k,1}, \psi_{k,1}), \dots, (W_{k,m(k)}, \psi_{k,m(k)})$

$\{W_{k,i} \mid k, i \in \mathbb{N}\}$ is regular: for this, we only need the loc. finality, everything else is automatic.

$\overline{V_k}$ is covered by fin. many V_j, each V_j intersects fin. many others, $W_{k,i} \subset \overline{V_k} \cap V_j$ for some j. $\Rightarrow$ $W_{k,i} \cap W_{j,i'} \neq \emptyset$ for fin. many j.
For each j there are fin. many $W_{j,i'}$. □

Def. $X = \{X_\alpha\}_\alpha$ open cover of smooth manifold M.
A partition of unity subordinate to X is a collection of smooth functions $\varphi_\alpha : M \to \mathbb{R}$, $\alpha \in A$ s.t.
1) $0 \leq \varphi_\alpha \leq 1$
2) $\operatorname{supp} \varphi_\alpha \subset X_\alpha$
3) $\{\operatorname{supp} \varphi_\alpha\}_{\alpha \in A}$ is locally finite
4) $\forall x \in M$: $\sum_{\alpha \in A} \varphi_\alpha(x) = 1$ (This is a finite sum by 2).)

Theorem. M smooth manifold, $X = \{X_\alpha\}_{\alpha \in A}$ open cover
$\Rightarrow$ $\exists$ partition of unity φ_α subordinate to X.

PROOF: $\{W_i\}_i$ regular refinement of X.

$U_i := \psi_i^{-1}(B_1(0))$, $V_i := \psi_i^{-1}(B_2(0))$

$$f_i := \begin{cases} H \circ \psi_i & \text{on } W_i \\ 0 & \text{on } M \setminus \overline{V_i}. \end{cases}$$ (On $W_i \setminus \overline{V_i}$ both definitions give 0.)

$\Rightarrow$ f_i is well-defined, smooth, $\operatorname{supp} f_i \subset W_i$

$g_i : M \to \mathbb{R}$, $g_i(x) := \dfrac{f_i(x)}{\sum_j f_j(x)}$ $\{W_i\}$ is loc. finite so the denominator has only fin. many terms.

$\Rightarrow g_i$ smooth.

($f_i \equiv 1$ on $U_i \ni x$ $\to$ $g_i(x)$ has nonzero denominator)

We have $0 \leq g_i \leq 1$ and $\sum_i g_i \equiv 1$.

$\forall i$ $\exists \alpha(i) \in A$ s.t. $W_i \subset X_{\alpha(i)}$. Let $\varphi_\alpha : M \to \mathbb{R}$, $\varphi_\alpha(x) = \sum_{i:\, \alpha(i)=\alpha} g_i(x)$.

This is a desired partition of unity. □

Corollary. M smooth manifold, $A \subset M$ closed, $U \supset A$ open $\Rightarrow \exists \varphi \in C^\infty(M)$ s.t. $\varphi|_A \equiv 1$ and $\operatorname{supp} \varphi \subset U$.

PROOF: $U_0 := U$, $U_1 := M \setminus A$

$\to$ $\{\varphi_0, \varphi_1\}$ partition of unity subordinate to the open cover $\{U_0, U_1\}$

$\varphi_1|_A \equiv 0$ $\Rightarrow$ $\varphi_0 =: \varphi$ is the desired function. □

We call such a φ a bump function as well.

Extension lemma. M smooth manifold. $A \subset M$ closed, $U \supset A$ open, $f \in C^\infty(A)$
$\Rightarrow \exists \tilde{f} \in C^\infty(M)$, $\tilde{f}|_A = f$, $\operatorname{supp} f \subset U$.

PROOF: f is smooth on A $\Rightarrow$ extends to a smooth function on $W \supset A$ open.

We may assume $W \subset U$ (or simply take $W \cap U$ instead of W).

Let φ be a bump function for A supported in W.

$$\tilde{f}(x) := \begin{cases} \varphi(x) \cdot f(x) & x \in W \\ 0 & x \in M \setminus \operatorname{supp} \varphi \end{cases}$$

This function is indeed a smooth extension. □

The condition of A being closed is necessary (exercise).

Global Analysis 1, lecture 3

17.10.2017

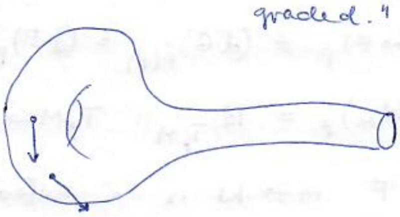
There was a mistake: Exercise 1.1. should have $\tau(x) = -\sigma(-x)$.

Today: [1] Ch. 3.

„I'm glad you found this new room, which is fancier and has entertainment on the walls, so we got upgraded."

Tangent vectors, tangent bundle

Think of tan. vectors as directional derivatives of functions on M.



Def. A linear map $V: C^\infty(M) \to \mathbb{R}$ is called a derivation at $p \in M$ if V satisfies the following:

$$V(f \cdot g) = f(p) \cdot V(g) + V(f) \cdot g(p) \quad \forall f, g \in C^\infty(M).$$

The tangent space at p is the set

$$T_p M = \{ V: C^\infty(M) \to \mathbb{R} \mid V \text{ is a derivation at } p \}.$$

It is easily seen that this is a linear space.

First we will just study these derivations for their own sake.

Lemma. $\forall p \in M$, $V \in T_p M$, $f, g \in C^\infty(M)$, we have:

a) If f is constant, then $V(f) = 0$.

b) If $f(p) = g(p) = 0 \Rightarrow V(f \cdot g) = 0$.

PROOF: a) $f = \lambda \cdot 1$, $\lambda \in \mathbb{R} \Rightarrow V(f) = V(\lambda \cdot 1) = \lambda \cdot V(1)$

$$V(1) = V(1 \cdot 1) = 1(p)\, V(1) + V(1) \cdot 1(p) = V(1) + V(1) \Rightarrow V(1) = 0.$$

b) $V(fg) = V(f)\, g(p) + f(p)\, V(g) = 0 + 0 = 0$ □

To get a geometric picture of this tangent space, we want to describe differentials of smooth maps between manifolds.

We always have diffeomorphisms U diffeomorphic to an open set in $\mathbb{R}^n$

→ we can use a local linear structure.

Def. Let $F: M \to N$ be a smooth map. The differential of F at p is the map $(dF)_p: T_p M \to T_{F(p)} N$ defined by $(dF_p)(V)(f) = V(f \circ F)$ for $f \in C^\infty(N)$, $V \in T_p M$.

$(dF)_p(V)$ is a derivation at $F(p)$:

$$(dF)_p(V)(f \cdot g) = V((f \cdot g) \circ F) = V((f \circ F) \cdot (g \circ F)) =$$

$$= V(f \circ F)\, (g \circ F)(p) + (f \circ F)(p)\, V(g \circ F) =$$

$$= (dF)_p(V)(f)\, g(F(p)) + f(F(p))\, (dF)_p(V)(g)$$

Proposition (Properties of the differential)

Let M, N, P be manifolds with or without boundary.

Suppose $F: M \to N$, $G: N \to P$ are smooth maps, let $p \in M$. Then:

a) $(dF)_p : T_pM \to T_{F(p)}N$ is linear

b) $d(G\circ F)_p = (dG)_{F(p)} \circ (dF)_p : T_pM \to T_{G(F(p))}P$ (chain rule)

c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.

d) If $F: M \to N$ is a difeomorphism, then $(dF)_p : T_pM \to T_{F(p)}N$ is a linear isomorphism with $(dF)_p^{-1} = (dF^{-1})|_{F(p)}$

PROOF: a), b), c) are instructive.

d) follows by combining the other three properties:

$$\mathrm{Id}_{T_{F(p)}M} = d(F\circ F^{-1})_{F(p)} = (dF)_p \circ (dF^{-1})_{F(p)}$$

$$\mathrm{Id}_{T_pM} = d(F^{-1}\circ F)_p = (dF^{-1})_{F(p)} \circ (dF)_p$$

□

The differential is a local operator in the following sense:

Proposition. Let M be a smooth manifold without boundary, $p \in M$, $V \in T_pM$.

Let $f, g \in C^\infty(M)$, and suppose $\exists U \supseteq p$ nbh. with $f|_U = g|_U$.

Then $V(f) = V(g)$.

PROOF: Let $h = f - g \Rightarrow h|_U \equiv 0$.

Take the bump function $\psi: M \to \mathbb{R}$ with $\psi \equiv 1$ on supp h and supp $\psi \subset M \setminus \{p\}$. (We know such a fuction exists.)

Then $h = h \cdot \psi$ and so

$$V(h) = V(h\cdot\psi) = h(p)\,V(\psi) + V(h)\cdot\psi(p) = 0+0=0.$$

And V is linear, hence $V(f) = V(g)$. □

Proposition. Let M be a smooth manifold with or without boundary and $U \subset M$ be some open set. We denote $i: U \hookrightarrow M$.

Then it holds that $\forall p \in U$ the differential $(di)_p : T_pU \to T_pM$ is an isomorphism.

(This is an important step towards our goal, which is to be able to look at the tangent space in $\mathbb{R}^n$ and not on M itself.)

Global Analysis I, lecture 3

PROOF: We only need to show that $(di)_p$ is bijective.

Injectivity: take $V \in T_p U$ with $(di)_p(V) = 0$.

Let B be a nbh.: $p \in B$, with $\overline{B} \subset U$.

For $f \in C^\infty(U)$ $\exists\, \tilde{f} \in C^\infty(M)$ such that $\tilde{f}|_{\overline{B}} = f|_{\overline{B}}$.

(They might differ on $U \setminus \overline{B}$, but we don't care, since only a nbh. of p is important.)

$\Rightarrow V(f) = V(\tilde{f}|_U)$ and $V(\tilde{f}|_U) = V(\tilde{f} \circ i) = (di)_p^{(U)}(\tilde{f}) = 0$.

Thus $V(f) = 0$. for all $f \in C^\infty(U)$ $\Rightarrow V = 0$. $\Rightarrow (di)_p$ is injective.

Surjectivity: let $W \in T_p M$ and define $V: C^\infty(U) \to \mathbb{R}$ by

$V(f) := W(\tilde{f})$ where $\tilde{f}: M \to \mathbb{R}$ with $\tilde{f}|_{\overline{B}} = f$.

Then V is a well-defined derivation at p.

On B we have for $g \in C^\infty(M)$: $g \circ i = \widetilde{g \circ i} = g$.

$\Rightarrow (di)_p(V)(g) = V(g \circ i) = W(\widetilde{g \circ i}) = W(g)$, so $(di)_p$ is surjective. □

In this way we identify $T_p U$ with $T_p M$.

Proposition. If M is n-dimensional, then $T_p M$ is an n-dimensional vector space $\forall p \in M$.

PROOF: Let (U, φ) be a smooth chart around $p \in U \subset M$.

Since $\varphi: U \to \varphi(U) \subset \mathbb{R}^n$ is a diffeomorphism onto its image,

we have $(d\varphi)_p : T_p U \xrightarrow{\sim} T_{\varphi(p)} \varphi(U)$ is an isomorphism.

The map $(di)_p : T_{\varphi(p)} \varphi(U) \xrightarrow{\sim} T_{\varphi(p)} \mathbb{R}^n$ is also an isomorphism,

so $T_p M \simeq T_p U \simeq T_{\varphi(p)} \varphi(U) \simeq T_{\varphi(p)} \mathbb{R}^n$.

For $a \in \mathbb{R}^n$, we have $T_a \mathbb{R}^n \cong \mathbb{R}^n$

Take $v = (v_1, \dots, v_n) \in \mathbb{R}^n$. The map

$$\begin{aligned} \mathbb{R}^n &\longrightarrow T_a \mathbb{R}^n \\ v &\longmapsto \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}\Big|_a \end{aligned} \qquad (*)$$

is clearly injective.

For surjectivity, observe that $f \in C^\infty(\mathbb{R}^n)$ can be written as

$$f(x) = f(a) + \underbrace{\sum_{i=1}^n \frac{\partial f(a)}{\partial x_i}(x_i - a_i)}_{(1)} + \underbrace{\sum_{i,j=1}^n (x_i - a_i)(x_j - a_j) \cdot \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(a + t \cdot (x-a))\, dt}_{(2)}.$$

(Taylor expansion)

For a derivation w at a, the term (2) gets annihilated, as does the constant term f(a).

So $w(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)\, w(x_i - a_i) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) \cdot w(x_i)$

So taking $v = (w(x_1), \dots, w(x_n))$ we get that(*) is surjective. □

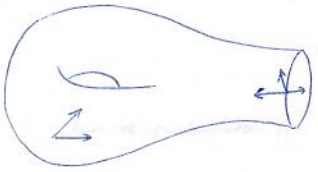
Note that the definition we gave is purely algebraic, so it can be actually used in computation.

<u>For manifolds with boundary</u>

<u>Lemma.</u> Let $i : \overline{\mathbb{H}}^n \hookrightarrow \mathbb{R}^n$ be the inclusion map ($\overline{\mathbb{H}}^n$ is the upper half-space). Then $(di)_a : T_a \overline{\mathbb{H}}^n \to T_a \mathbb{R}^n$ is an isomorphism.

PROOF: When $a \in \text{int}\, \overline{\mathbb{H}}^n$: previous proof works.

When $a \in \partial \overline{\mathbb{H}}^n$, then use that by definition $f : \overline{\mathbb{H}}^n \to \mathbb{R}^n$ is smooth if it admits a smooth extension to some open set $U \supset \overline{\mathbb{H}}^n$. Then work with that extension. □



→ We have a full 2-dimensional tangent space on the boundary as well; the tangent space is determined by the interior, not the boundary.

<u>Corollary.</u> For an n-dim manifold M with boundary, $T_pM \simeq \mathbb{R}^n$ $\forall p \in M$. □

<u>Example.</u> Tangent space to a vector space.

Let V be a vector space of dimension n, then the choice of basis $e_1, \dots, e_n$ gives a map $\psi : \mathbb{R}^n \to V$, $x \mapsto \sum_{i=1}^{n} x_i e_i$ and so we have a topology on V.

(The top. is independent from the choice of the basis.)

$\varphi := \psi^{-1}$ can be viewed as a chart → this gives a smooth structure, again indep. from $\{e_i\}_i$, because any two basis differ by a matrix in $GL(n, \mathbb{R})$.

<u>Proposition</u> (Tangent space to a vector space)

Let V be a vector space. $\forall a \in V$ the map $V \to T_a V$, $v \mapsto D_v|_a$ (where $D_v|_a f = \frac{d}{dt}\big|_{t=a} f(a+tv)$, $\forall f \in C^\infty(V)$) is a canonical isomorphism.

(So this is also basis-independent.)

Global Analysis 1, lecture 3 – 4

Moreover, for a linear map $L : V \to W$ between vector spaces, the diagram

$$\begin{tikzcd} V \arrow[r, "\sim"] \arrow[d, "L"'] & T_a V \arrow[d, "(dL)_a"] \\ W \arrow[r, "\sim"] & T_{L(a)} W \end{tikzcd}$$

commutes.

PROOF: For the first statement, one must choose a basis and work with that. (PO).

For the diagram, L is smooth and its matrix coefficients are smooth functions wrt a choice of a basis.

Then $(dL)_a \left(D_v|_a \right) f = D_v|_a (f \circ L) = \frac{d}{dt}\Big|_{t=0} f(L(a+tv)) =$

$$= \frac{d}{dt}\Big|_{t=0} f(La + tLv) = D_{Lv}\big|_{La} f$$

☐

The tangent bundle

Consider the following set: $TM := \bigsqcup_{p \in M} T_p(M)$

The disjoint union ($\sqcup$) allows us to keep track of at which point p we got the tangent vector.

Write (p, v) or v_p for the elements of TM.

The projection map. $\pi : TM \to M$
$(p, v) \mapsto p$

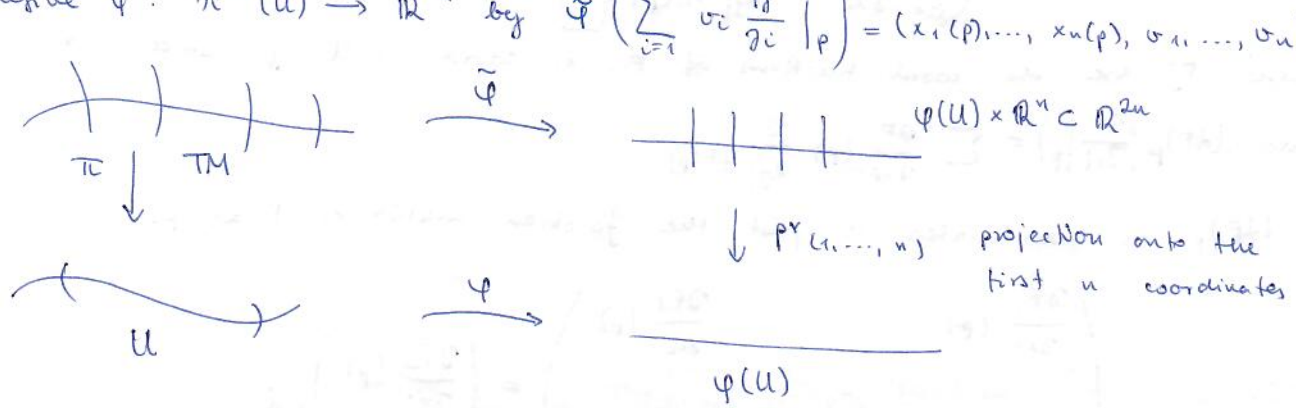
Prop. If M is a smooth n-manifold, then TM has a natural topology and smooth structure, making it into a $2n$-manifold and the projection map π is smooth.

PROOF: Let (U, φ) be a smooth chart for M.

Consider the set $\pi^{-1}(U) = \{ (p, v) \mid p \in U,\ v \in T_p M \}$.

If $x_1, \dots, x_n$ are the coordinate functions of $\varphi : U \to \mathbb{R}^n$,

define $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by $\tilde{\varphi}\left(\sum_{i=1}^{n} v_i \frac{\partial}{\partial i}\Big|_p \right) = (x_1(p), \dots, x_n(p), v_1, \dots, v_n)$.



Then $\tilde{\varphi}(\pi^{-1}(U)) = \varphi(U) \times \mathbb{R}^n$, which is open and $\tilde{\varphi}$ has an inverse

$$\tilde{\varphi}^{-1}(x_1, \dots, x_n, v_1, \dots, v_n) = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}\Big|_{\varphi^{-1}(x)}$$

$\Rightarrow$ $\tilde{\varphi}$ is a bijection $\Rightarrow$ TM is a topological manifold.

Sidetrack: computations in coordinates 18.10.2017

M smooth manifold, (U, φ) chart around p

$\varphi : U \to \varphi(U)$ diffeomorphism

$T_pU \xrightarrow{\sim} T_{\varphi(p)}\varphi(U) \xrightarrow{\sim} T_{\varphi(p)}\mathbb{R}^n$

$\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\right)_{i=1}^{n}$ form a basis for $T_{\varphi(p)}\mathbb{R}^n$

We write $\frac{\partial}{\partial x_i}\Big|_p := (d\varphi)_p^{-1}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\right)$ (this is relative to the chart (U, φ))

$$= d(\varphi^{-1})_{\varphi(p)}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\right)$$

By construction, $\left(\left(\frac{\partial}{\partial x_i}\right)_p\right)_{i=1}^{n}$ form a basis of T_pU.

If $f \in C^\infty(M)$, then $\frac{\partial}{\partial x_i}\Big|_p f = \frac{\partial}{\partial x_i}\Big|_{\varphi(p)}(f \circ \varphi^{-1}) = \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}(\varphi(p)) =$

$$= \left(\frac{\partial \hat{f}}{\partial x_i}(\hat{p})\right) \quad \text{where } \hat{f} = f \circ \varphi^{-1},\ \hat{p} = \varphi(p).$$

Prop. If M is a smooth manifold (with or without boundary), then T_pM is an n-dim vector space and for any smooth chart (U, φ) around p, $\left(\frac{\partial}{\partial x_i}\Big|_p\right)_{i=1}^{n}$ is a basis for T_pM.

Hence any $v \in T_pM$ can be written $v = \sum v_i \frac{\partial}{\partial x_i}\Big|_p$, where $v_i = v(x_i)$, x_i is the i^{th} coordinate function wrt. (U, φ).

For $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ open, $F : U \to V$ smooth map; $p \in U$, $f \in C^\infty(V)$:

$$(dF)_p\left(\frac{\partial}{\partial x_i}\Big|_p\right)(f) = \frac{\partial}{\partial x_i}\Big|_p (f \circ F) = \sum_{j=1}^{m} \frac{\partial}{\partial y_j}(F(p)) \cdot \frac{\partial F_j}{\partial x_i}(p) =$$

$$= \left(\sum_{j=1}^{m} \frac{\partial F_j}{\partial x_i}(p) \frac{\partial}{\partial y_j}\Big|_{F(p)}\right)(f)$$

where F_j are the coord. functions of F, x_i coord. in U, y_j coord. in V.

Thus $(dF)_p\left(\frac{\partial}{\partial x_i}\Big|_p\right) = \sum \frac{\partial F_j}{\partial x_i}(p) \frac{\partial}{\partial y_j}\Big|_{F(p)}$

So $(dF)_p$ in coordinates is just the Jacobian matrix of F at p.

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1}(p) & \cdots & \frac{\partial F_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1}(p) & \cdots & \frac{\partial F_m}{\partial x_n}(p) \end{pmatrix} = \left(\frac{\partial F_i}{\partial x_j}(p)\right)_{i,j}$$

Global Analysis 1, Lecture 4

For · $F: M \to N$ smooth map between manifolds,

(U, φ) chart around p,

(V, ψ) chart around $F(p)$,

we get the coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$, $\hat{p} = \varphi(p)$.

Hence $(d\hat{F})_{\hat{p}} = \left(\frac{\partial \hat{F}_i}{\partial x_j} (\hat{p}) \right)_{i,j}$

Since $F\varphi^{-1} = \psi^{-1}\hat{F}$, we find $(dF)_p \left(\frac{\partial}{\partial x_i}\Big|_p \right) = (dF)_p \, (d\varphi^{-1})_{\hat{p}} \left(\frac{\partial}{\partial x_i}\Big|_{\hat{p}} \right) =$ by unfolding definitions. Now we use the chain rule

$$= d(F \circ \varphi^{-1})_{\hat{p}} \left(\frac{\partial}{\partial x_i}\Big|_p \right) = d(\psi^{-1})_{\hat{F}(\hat{p})} \left(d\hat{F}_{\hat{p}} \left(\frac{\partial}{\partial x_i}\Big|_{\hat{p}} \right) \right) =$$

$$= d(\psi^{-1})_{\hat{F}(\hat{p})} \left(\sum_j \frac{d\hat{F}_j}{dx_i}(\hat{p}) \frac{\partial}{\partial y_j}\Big|_{\hat{F}(\hat{p})} \right) = \sum \frac{\partial \hat{F}_j}{\partial x_i}(\hat{p}) \left(\frac{\partial}{\partial y_j}\Big|_{F(p)} \right)$$

So $(dF)_p$ in coordinates is represented by the Jacobian matrix of $\hat{F}$.

Change of coordinates: (U, φ) chart around p, (V, ψ) chart around p,

x_i coordinates w.r.t. φ, $\frac{\partial}{\partial x_i}\Big|_p \in T_pM$,

y_j coordinates w.r.t. ψ, $\frac{\partial}{\partial y_j}\Big|_p \in T_pM$

Write the transition map $\psi \circ \varphi^{-1}$ as $\psi \circ \varphi^{-1}(x) = (y_1(x), \dots, y_n(x))$.

$$d(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)} \right) = \sum_j \frac{\partial y_j}{\partial x_i}(\varphi(p)) \frac{\partial}{\partial y_j}\Big|_{\psi(p)}.$$

$$\Rightarrow \frac{\partial}{\partial x_i}\Big|_p = d(\varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)} \right) = (d\psi^{-1})_{\varphi(p)} \circ d(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)} \right).$$

$$= d(\psi^{-1})_{\psi(p)} \left(\sum_j \frac{\partial y_j}{\partial x_i}(\varphi(p)) \frac{\partial}{\partial y_j}\Big|_{\psi(p)} \right)$$

$$= \sum_j \frac{\partial y_j}{\partial x_i}(\hat{p}) \frac{\partial}{\partial y_j}\Big|_p$$

Therefore $\left(d(\psi \circ \varphi^{-1})(v) \right)_j = \sum_j \frac{\partial y_j}{\partial x_i}(\hat{p}) \, v_j \quad \forall v \in T_p(\mathbb{R}^n)$

Now we return to proving that the tangent bundle is a manifold.

(U, φ) chart around $p \in M$

$\pi^{-1}(U) = \{ (p, v) \mid p \in U,\ v \in T_pM \}$

$\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$

$$\sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}\Big|_p \mapsto (\varphi_1(p), \dots, \varphi_n(p), v_1, \dots, v_n)$$

$$(x_1(p), \dots, x_n(p), v_1, \dots, v_n)$$

We have shown that $\tilde{\varphi}$ is a bijection between $\pi^{-1}(U)$ and $\varphi(U) \times \mathbb{R}^n$.

Choosing a cover of M by charts U_i, $\{\tilde{\varphi}_i\}$ makes TM into a top. manifold.

We wish to show that this gives a smooth structure.

Let (U, φ), (V, ψ) be smooth charts for M, $U \cap V \neq \emptyset$.

$$\tilde{\varphi}\left(\pi^{-1}(U) \cap \pi^{-1}(V)\right) = \varphi(U \cap V) \times \mathbb{R}^n$$

$$\tilde{\psi}\left(\pi^{-1}(U) \cap \pi^{-1}(V)\right) = \psi(U \cap V) \times \mathbb{R}^n$$

The transition map $\tilde{\psi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$ can be written as follows:

$$\tilde{\psi} \circ \tilde{\varphi}^{-1}(x_1(p), \dots, x_n(p), v_1, \dots, v_n) = \left(y_1(p), \dots, y_n(p), \sum_j \frac{\partial y_1}{\partial x_j}\Big|_{\varphi(p)} v_j, \dots, \sum_j \frac{\partial y_n}{\partial x_j}\Big|_{\varphi(p)} v_j \right)$$

This is smooth because the charts U, V are smoothly compatible. $\square$

The map $\pi : TM \to M$ is smooth because in the charts $(\pi^{-1}(U), \tilde{\varphi})$ it is given by projection on the last n-coordinate, which is a smooth map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$.

Def: The global differential of a smooth map $F : M \to N$ is the map $dF : TM \to TM$ given by $(dF)(v_p) = (dF)_p(v_p)$.

Prop: The global differential $dF : TM \to TM$ is a smooth map.

PROOF: $dF(x_1, \dots, x_n, v_1, \dots, v_n) = \left(F_1(x), \dots, F_n(x), \sum_i \frac{\partial F_1}{\partial x_i} v_i, \dots, \sum_i \frac{\partial F_n}{\partial x_i} v_i \right)$, which is smooth because F is smooth. $\square$

(All these proofs run on the same idea.)

Properties of the global differential

1) $d(G \circ F) = dG \circ dF$ $(F : M \to N,\ G : N \to P)$
2) $d(\mathrm{Id}_M) = \mathrm{Id}_{TM}$
3) $F : M \to N$ diffeom. $\Rightarrow$ $dF : TM \to TM$ diffeom. with $(dF)^{-1} = d(F^{-1})$.

Global Analysis 1, lecture 4-5

Vector bundles

Def. Let M be a top. space. A rank k vector bundle over M is a topological space E together with a continuous map $\pi: E \to M$ such that

1) $\forall p \in M$ the fiber $E_p = \pi^{-1}(p)$ is a k-dimensional real vector space.
2) $\forall p \in M$ has a nbh. U and $\exists\, \Phi: \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ homeomorphism. s.t. $\pi_U \circ \Phi = \pi$ where $\pi_U : U \times \mathbb{R}^k \to U$, $(u,v) \mapsto u$. (So Φ respects the fiber projection).
3) $\forall q \in U$ the restriction

$$\begin{tikzcd} \Phi : \pi^{-1}(q) \arrow[r] \arrow[d, equal] & \{q\} \times \mathbb{R}^k \\ E_q \arrow[ur] & \end{tikzcd}$$

is a linear map.

If E, M are smooth manifolds and π, Φ are smooth, then E is a smooth vector bundle over M.

Example. $TM \to M$ is a smooth vector bundle.

Remark. One defines complex vector bundles in an analogous way.

Terminology. E is the total space of the vector bundle,
M is the base,
π is the bundle projection, Φ is a local trivialisation.

Example. The trivial rank k vector bundle. is $E := M \times \mathbb{R}^k \longrightarrow M$, $(m,v) \longmapsto m$

(This also works in the complex case.)

Lemma. Let $\pi: E \to M$ be a smooth rank k vector bundle.
Suppose $\Phi: \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi: \pi^{-1}(V) \to V \times \mathbb{R}^k$ are local trivialisations with $U \cap V \neq \emptyset$. Then $\exists$ a smooth map $\tau: U \cap V \to GL(k,\mathbb{R}) \subset M_{k\times k}(\mathbb{R}) \cong \mathbb{R}^{k^2}$
(note that $GL(k,\mathbb{R})$ is open in $M_{k\times k}(\mathbb{R})$, so it is a smooth manifold.),
such that $\Phi \circ \Psi^{-1}(p,v) = (p, \tau(p)\circ v)$ where $\tau(p)\circ v$ is given by matrix multiplication.
(Note that $\Phi \circ \Psi^{-1}: U \cap V \times \mathbb{R}^k \longrightarrow U \cap V \times \mathbb{R}^k$.)

PROOF: By definition, the following diagram commutes.

$$\begin{tikzcd} (U\cap V) \times \mathbb{R}^k & \pi^{-1}(U\cap V) \arrow[l, "\Psi"'] \arrow[r, "\Phi"] \arrow[d, "\pi"] & (U \cap V) \times \mathbb{R}^k \arrow[dl, "\pi_{U\cap V}"] \\ & U \cap V \arrow[from=1-1, "\pi_{U\cap V}"'] & \end{tikzcd}$$

$\Rightarrow \exists\, \sigma: U \cap V \times \mathbb{R}^k \to \mathbb{R}^k$ smooth map, $\Phi \circ \Psi^{-1}(p,v) = (p, \sigma(p,v))$

For fixed $p \in U \cap V$ the map $v \mapsto \sigma(p,v)$ is a linear isomorphisms because of property 3) in the def.

$\Rightarrow \exists\, \tau(p) \in GL(k,\mathbb{R})$ with $\sigma(p,v) = \tau(p) \circ v = \sum_{ij} v_i\, \tau(p)_{ij}\, e_j$

w.r.t. the standard basis e_j of $\mathbb{R}^k$.

Since σ is smooth, it follows that $p \mapsto \tau_{ij}(p)$ are smooth maps. $\square$

This process can be reversed in order to obtain vector bundles.

24.10.2017

Lemma: (Chart lemma) Let M be a smooth manifold with or without boundary. Suppose that $\forall p \in M$ we are given a vector space E_p, $\dim E_p = k$ (constant)

Define $E := \coprod_{p \in M} E_p$, and $\pi : E \to M$
$$e_p \mapsto p$$

If we are given the following:

1) $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M.

2) $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ bijections with V a fixed vector space, $\dim V = k$.

such that the restrictions

$\Phi_\alpha \; E_p = \pi^{-1}(p) \longrightarrow \{p\} \times V$ are vector space isomorphisms

3) $\forall \alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$. a smooth map $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(V)$.

such that $\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times V \longrightarrow (U_\alpha \cap U_\beta) \times V$ has the form

$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha,\beta}(p) \cdot v)$.

Then E has a unique topology and smooth structure, making it into a manifold (with or without boundary) and $E \xrightarrow{\pi} M$ is a smooth rank k vector bundle over M, with local trivialisations $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in A}$.

PROOF: Very similar to the concrete case we did before, we consider it to be standard (and usefull), and it can be looked up in Lee's book. □

Examples. The cotangent bundle T^*M (* stands for dual)

$E_p := (T_p M)^* = \{ \varphi : T_p M \to \mathbb{R} \mid \varphi \text{ lineear} \} =: T_p^* M$

$E := \coprod_{p \in M} E_p$, for a chart (U_i, φ_i) of M we define

$\Phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n$

$\sum_{i=1}^{n} v_i dx_i|_p \longmapsto (p, v_1, \dots, v_n)$ $\quad n = \dim M$

where $(dx_i|_p)$ is the dual basis to $\left(\frac{\partial}{\partial x_i}\Big|_p\right)$, i.e. $dx_i|_p \left(\frac{\partial}{\partial x_j}\Big|_p \right) = \delta_{ij}$

If (U_j, φ_j) is another chart with $U_i \cap U_j \neq \emptyset$ with coordinates y_j, then $\Phi_i \circ \Phi_j^{-1}(p, v_1, \dots, v_n) = \left(p, \frac{\partial y_j}{\partial x_1}(p) v_j, \dots, \frac{\partial y_j}{\partial x_n}(p) v_j\right)$

which is a smooth map $U_i \cap U_j \times \mathbb{R}^n \longrightarrow U_i \cap U_j \times \mathbb{R}^n$ as

$p \longmapsto \left(\left(\frac{\partial y_j}{\partial x_i}(p) \right) \right)_{i,j} \in GL(n, \mathbb{R})$ $\longrightarrow$ we can apply the chart lemma.

Global Analysis I, lecture 5

Example. The bundle of alternating k-tensors is defined to be

$$\Lambda^k T^* M := \coprod_{p \in M} \Lambda^k T_p^* M .$$

Let (x_i) be local coordinates at p and $\omega \in \Lambda^k T^* M$.

Then ω looks like $\omega = \sum_I v_I \, dx_I$ where $I = (i_1, \dots, i_k)$ is an increasing multiindex $(i_1 < \dots < i_k)$, $dx_I = dx_{i_1}|_p \wedge \dots \wedge dx_{i_k}|_p$.

This gives the maps

$$\psi : \pi^{-1}(U) \longrightarrow U \times \Lambda^k \mathbb{R}^n$$
$$\omega_p \longmapsto (p, \sum v_I e_I) \qquad e_I = e_{i_1} \wedge \dots \wedge e_{i_k},$$

where e_j is a basis for $\mathbb{R}^n$.

The functions $\tau : U \cap V \longrightarrow GL(n, R)$ defined from $T^* M$ will be used to define $\tau_k : U \cap V \to GL(\Lambda^k \mathbb{R}^n)$ via

$$\tau_k(p)(v_1 \wedge \dots \wedge v_k) = \tau(p) v_1 \wedge \dots \wedge \tau(p) v_k$$

This structure defines $\Lambda^k T^* M$.

Notation. $\Lambda^* T^* M := \bigoplus_{k \geq 0} \Lambda^k T^* M$

The direct sum bundle $E_1 \oplus E_2$ has transition maps $\tau = \tau_1 \oplus \tau_2 = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$.

Def. Let $\pi : E \to M$ be a smooth vector bundle. A section of E is a map $s : M \to E$ s.t. $\pi \circ s = id_M$.

A continuous section is a section that is cont.,

a smooth section is a section that is smooth.

Notation. $\Gamma(M, E) = \{ s : M \to E \mid \text{cont. section} \}$

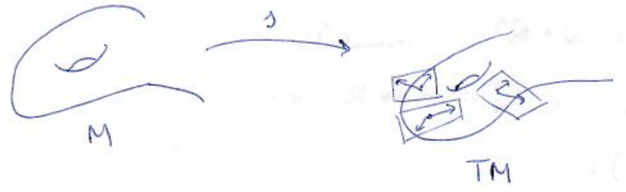
$\Gamma^\infty(M, E) = \{ s ; M \to E \mid \text{smooth section} \}$.

Def. A vector field X is an element of $\Gamma^\infty(M, TM) =: \mathfrak{X}(M)$,

a covector field or differential 1-form is a section $\omega \in \Gamma^\infty(M, T^* M) =: \Omega^1(M)$.

A differential k-form is a section $\omega \in \Gamma^\infty(M, \Lambda^k T^* M) =: \Omega^k(M)$.

Def. A local section is a map $s : U \to E$ with $\pi \circ s = id_U$ for some $U \subset M$ open.



Local frame

$E \xrightarrow{\pi} M$ smooth vector bundle

$U \subset M$ open set

Def. A k-tuple $(\sigma_1, \dots, \sigma_k)$ is a local frame if σ_i are smooth sections over U, and $\forall p \in U$: $(\sigma_1(p), \dots, \sigma_k(p))$ form a basis of $E_p = \pi^{-1}(p)$.

Example. (Frames and trivialisations)

Let $\Phi: \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ be a trivialisation.

Define $\sigma_i: U \longrightarrow \pi^{-1}(U)$, where e_i are the standard basis on $\mathbb{R}^k$, $\sigma_i(u) = \Phi^{-1}(u, e_i)$.

$$\begin{tikzcd} u \arrow[d, mapsto] & U \arrow[r] \arrow[d] & \pi^{-1}(U) \\ (u, e_i) & U \times \mathbb{R}^k \arrow[ur, "\Phi^{-1}"] & \end{tikzcd}$$

σ_i is smooth because Φ is a diffeomorphism.

$(\sigma_1, \dots, \sigma_k)$ is called the local frame associated with the trivialisation Φ.

Prop. Any smooth local frame for $E \xrightarrow{\pi} M$ over U is associated with a trivialisation $\Phi: \pi^{-1}(U) \to U \times \mathbb{R}^k$ over U.

PROOF: Let $(\sigma_1, \dots, \sigma_k)$ be a local frame over U.

Define
$$\Psi: U \times \mathbb{R}^k \longrightarrow \pi^{-1}(U)$$
$$(p, v) \longmapsto \sum_{i=1}^{k} v_i \sigma_i(p)$$

This Ψ is bijective since (σ_i) is a frame, and $\sigma_i(p) = \Psi(p, e_i)$ where e_i is the standard basis of $\mathbb{R}^k$.

So algebraically, we are clear. We nts that Ψ is a local diffeomorphism.

For $p \in U$ choose a nbh. V of p s.t. $V \subseteq U$, and there is a trivialisation $\psi: \pi^{-1}(V) \longrightarrow V \times \mathbb{R}^k \subseteq U \times \mathbb{R}^k$.

It suffices to show that $\Phi \circ \Psi$ is a diffeomorphism because Φ is a diffeom.

$$\begin{tikzcd} V \times \mathbb{R}^k \arrow[r, "\Psi"] \arrow[dr, "pr_V"'] & \pi^{-1}(V) \arrow[r, "\Phi"] \arrow[d, "\pi"] & V \times \mathbb{R}^k \arrow[dl, "pr_V"] \\ & V & \end{tikzcd}$$

Now the composite $\Phi \circ \sigma_i : V \longrightarrow V \times \mathbb{R}^k$ is smooth.

$\Rightarrow$ there are smooth maps $\sigma_i^1, \dots, \sigma_i^k : V \to \mathbb{R}$ s.t.

$$\Phi \circ \sigma_i = (p, \sigma_i^1(p), \dots, \sigma_i^k(p))$$

and
$$\Phi \circ \Psi\,(p, x) = \Phi\Big(\sum_i v_i \sigma_i(p)\Big) = \sum_i v_i \Phi(\sigma_i(p)) =$$
$$= \Big(p, \sum_i v_i \sigma_i^1(p), \dots, \sum_i v_i \sigma_i^k(p)\Big)$$
which is smooth.

Global Analysis I, lecture 5 – 6

(This proposition is the vector bundle analogue of the choice of a basis.)

To see that $(\Phi \circ \Psi)^{-1}$ is smooth, we observe that $\left(\sigma_i^j(p)\right)_{i,j}$ is an invertible matrix.

Let $\left(\tau_i^j(p)\right)_{i,j}$ be its inverse (inverse in $GL(k, \mathbb{R})$ is smooth).

So $\tau_i^j(p)$ depends smoothly on p.

$\Rightarrow \ (\Phi \circ \Psi)^{-1}(p, (w_1, \dots, w_n)) = \left(p, \left(\sum_i w_i \tau_i^j(p)\right)_j\right)$, which is smooth. □

Corollary. Suppose $(\sigma_1, \dots, \sigma_k)$ is a frame for $E \xrightarrow{\pi} M$ defined on all of M.

Then $E \cong M \times \mathbb{R}^k$ as vector bundles (iso.), i.e.

$$\begin{tikzcd} E \arrow[rr, "\simeq"] \arrow[dr, "\pi"'] & & M \times \mathbb{R}^k \arrow[dl, "pr_M"] \\ & M & \end{tikzcd}$$

the iso is compatible with the projections.

PROOF: The proof of the prop. directly gives us this. □

Corollary. Let (V, φ) be a smooth chart on M with coordinates (x_i) and (σ_j) is a frame for $E \xrightarrow{\pi} M$ defined V.

Then $\tilde{\varphi} : \pi^{-1}(V) \to \varphi(V) \times \mathbb{R}^k \subset \mathbb{R}^{n+k}$

$$\sum v_i \sigma_i(p) \mapsto (x_1(p), \dots, x_n(p), v_1, \dots, v_k)$$

is a smooth chart for E.

PROOF: $\tilde{\varphi} = (\varphi \times id_{\mathbb{R}^k}) \circ \Phi$.

(Suppose Φ is associated to (σ_j).) □

Proposition. Let $\pi : E \to M$ be a smooth vector bundle and $\tau : M \to E$ a section and (σ_j) a smooth local frame over $U \subset M$.

Then τ is smooth over U $\Leftrightarrow$ the coordinate fuctions $\tau_i : U \to \mathbb{R}$ defined by $\tau(p) = \left(p, \sum_i \tau_i(p) \sigma_i(p)\right)$ are smooth.

PROOF: Let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be the trivialisation assoc. with (σ_i).

Φ is a difeomorpism. So τ is smooth $\Leftrightarrow$ $\Phi \circ \tau : U \to U \times \mathbb{R}^k$ is smooth.

Now $\Phi \circ \tau(p) = (p, \tau_1(p), \dots, \tau_k(p))$ so the statement follows. □

Prop. (Smoothness of covector fields)

Let $\omega : M \to T^*M$. Then the following are equivalent:

a) ω is smooth

b) in every chart the component functions of ω are smooth.

c) every point is contained in some chart for which the component functions of ω are smooth.

d) for every smooth vector field $X \in \Gamma^\infty(M, TM)$ the function $\omega(X)$ is smooth.

e) for every open set and smooth vector field $X : U \to TM$ the function $\omega(X)$ is smooth on U.

$\omega(X)(p) = \omega_p(X_p) \in \mathbb{R} \quad \Rightarrow \omega(X) : M \to \mathbb{R}$ indeed.

Moreover, a vector field $X \in \Gamma^\infty(M, TM)$ acts on $C^\infty(M)$ via

$X(f)(p) = (X_p f)(p)$.

Xf is smooth when X is smooth.

The only "hard" implication is d) $\Rightarrow$ e).

$\forall p \in U$ choose a bump function χ with $\chi \equiv 1$ on some nbh. V of p and $\operatorname{supp} \chi \subset U$.

Then we set $\widetilde{X} = \chi \cdot X : M \to TM$ is a global smooth vector field, and $\widetilde{X}(f) = X(f)$ on V. So $X(f)$ is smooth on U since $p \in U$ was arbitrary.

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Notation. Vectors fields are denoted by X and $\mathfrak{X}(M) = \Gamma^\infty(M, TM)$, $X \in \mathfrak{X}(M)$

A tangent vector v_p is defined to be a derivation at p, $v_p : C^\infty(M) \to \mathbb{R}$,

$v_p(fg) = f(p)\, v_p(g) + v_p(f)\, g(p)$.

A vector field $X \in \mathfrak{X}(M)$ defines a map $X : C^\infty(M) \to$ (functions on M) by

$(Xf)(p) = (X_p f)$

Prop. Let $X : M \to TM$ be a section. TFAE:

a) X is smooth

b) $\forall f \in C^\infty(M) : \; Xf \in C^\infty(M)$

c) $\forall U \subseteq M$ open, $f \in C^\infty(U) : \; Xf \in C^\infty(U)$.

Pf: a) $\Rightarrow$ b) Assume that X is smooth, take $f \in C^\infty(M)$.

$p \in M$, choose coordinates x_i giving a loc. frame $\frac{\partial}{\partial x_i}\big|_p$ at p.

Then for x in a nbh. of p we have $Xf(x) = \sum X_i(x) \cdot \frac{\partial f}{\partial x_i}\big|_x$,

where X_i is the ith coord. function of X.

The component function X_i are smooth since X is smooth and projections are smooth. Hence Xf is smooth.

b) => c) $\tilde{X} := \chi \cdot X$, χ a bump function around $p \in U$, $\chi|_V \equiv 1$ for V nbh., supp $\chi \subset U$. $\Rightarrow \tilde{X}(f) = X(f)$ at $p \in U$ and thus Xf is smooth on U. ☐

c) => a) For coordinates (x_i) around $p \in M$, each x_i gives a smooth function $x_i : U \to \mathbb{R}$ on the coordinate nbh. U, $U \ni p$.

According to c), $X(x_i)$ is smooth

and $X(x_i) = \sum_j X_j(x) \frac{\partial x_i}{\partial x_j} = X_i(x)$. ☐

So vector fields induce maps $C^\infty(M) \to C^\infty(M)$ which are derivations $X(fg) = (Xf) \cdot g + f \cdot (Xg)$.

The converse is also true: all derivations arise in such a way.

<u>Prop.</u> M a smooth manifold with or without boundary, $D : C^\infty(M) \to C^\infty(M)$ a linear map satisfying $D(f \cdot g) = D(f) \cdot g + f \cdot D(g)$ $\forall f, g \in C^\infty(M)$.

$\to \exists X$ vector field s.t. $D(f) = X(f)$ $\forall f \in C^\infty(M)$.

Pf: Define $X_p f := (Df)(p)$.

Then $X_p(fg) = (\mathcal{D}f)(p) = f(p)(\mathcal{D}g)(p) + (\mathcal{D}f)(p) \cdot g(p) = f(p) X_p(g) + X_p(f) g(p)$

So X_p is indeed a derivation at p, and this defines a section $X : M \to TM$. This section X is smooth since $Xf = Df$ is smooth. ☐

Lie bracket of vector fields

If $X, Y : C^\infty(M) \to C^\infty(M)$ are derivations,

then $X \circ Y(f) = X(Yf)$ is <u>not</u> a derivation. (We get some second-order Leibniz rule.)

However, the <u>commutator</u> <u>$[X, Y] := (X \circ Y - Y \circ X)$</u> ("X bracket Y", "X commutator Y") is a derivation.

$$[X, Y](fg) = X(Y(fg)) - Y(X(fg)) =$$

$$= X(Y(f) \cdot g + f \cdot Y(g)) - Y(X(f) \cdot g + f \cdot X(g)) =$$

$$= XY(f) \cdot g + \underline{Yf \cdot Xg} + \underline{Xf\ Yg} + f \cdot XY(g) - YX(f)g - \underline{Xf \cdot Yg} -$$

$$- \underline{Yf \cdot Xg} - f \cdot YX(g) = XY(f) \cdot g - YX(f) \cdot g + f \cdot XY(g) - f \cdot YX(g) =$$

$$= [X, Y](f) \cdot g + f \cdot [X, Y](g).$$

Differential of a function $f \in C^\infty(M)$

Let $v \in T_pM$, $(df)_p(v) := v_p(f) \Rightarrow (df)_p \in T_p^*M$

df is the differential of f, and is a section of the cotangent bundle $M \to T^*M$.

Prop. $f \in C^\infty(M) \Rightarrow (df)$ is a smooth covector field. (i.e. a smooth section).

Pf: $(df)_p$ is clearly linear in v and for a vector $X \in \mathfrak{X}(M) = \Gamma^\infty(M, TM)$ we have $(df)(X) = Xf$ is smooth $\Rightarrow df$ is smooth. □

On a chart (U, φ) with coordinates (x_i):

$$df\Big|_p = \sum_i \frac{\partial f}{\partial x_i}\Big|_p dx_i \quad \text{is the usual 1-form.}$$

Also note that viewing $x_i : U \to \mathbb{R}$ as smooth functions, we obtain $dx_i|_p = dx_i|_p$ (so we defined things in a right way; remember that a priori dx_i was the basis dual to $\frac{\partial}{\partial x_i}$).

Def. A differential k-form on M is a smooth section $\omega : M \to \Lambda^k T^*M$.

We write $\Omega^k(M) = \Gamma^\infty(M, \Lambda^k T^*M)$ and $\Omega^*(M) = \bigoplus_{k \geq 0} \Omega^k(M)$.

(If $k > \dim M$, $\Omega^k(M)$ is zero, so this really is a finite direct sum.)

Wedge product of forms

$\omega \in \Omega^k(M)$, $\eta \in \Omega^n(M)$ arbitrary forms

$X_1, \dots, X_{n+k} \in \mathfrak{X}(M)$ vector fields

$$(\omega \wedge \eta)(X_1, \dots, X_{n+k}) := \frac{1}{n!\,k!} \sum_{\sigma \in S_{n+k}} \mathrm{sgn}(\sigma) \cdot \omega(X_{\sigma(1)}, \dots, X_{\sigma(k)})\, \eta(X_{\sigma(k+1)}, \dots, X_{\sigma(n+k)})$$

The wedge product satisfies:

- $(\lambda \omega_1 + \mu \omega_2) \wedge \eta = \lambda \omega_1 \wedge \eta + \mu \omega_2 \wedge \eta$ $\quad \lambda, \mu \in \mathbb{R}$ $\quad$ distributive
- $\omega \wedge \eta = (-1)^{|\omega| \cdot |\eta|} \eta \wedge \omega$ where $|\omega| = k$ if $\omega \in \Omega^k(M)$. anticommutative
- for 1-forms $\omega_1, \dots, \omega_k$ and vector fields $X_1, \dots, X_k$ we have $(\omega_1 \wedge \dots \wedge \omega_k)(X_1, \dots, X_k) = \det(\omega_i(X_j))_{i,j}$

Def: Let $F : M \to N$ be a smooth map between manifolds M, N.

Suppose $\omega \in \Omega^k(N)$.

Define $F^*\omega \in \Omega^k(M)$ by setting $(F^*\omega)(X_1, \dots, X_k) = \omega(dF(X_1), \dots, dF(X_k)) =$
$= (\omega \circ dF)(X_1, \dots, X_k)$

$F^*\omega$ is the pullback of ω under F.

Global Analysis 1, lecture 6

Lemma. (Properties of the pullback) Let $F: M \to N$ be smooth

a) $F^*: \Omega^k(N) \to \Omega^k(M)$ is linear

b) $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$

c) In any smooth chart on N, with coordinates (y_i) we have

$$F^*\left(\sum_I \omega_I \, dy_{i_1} \wedge \dots \wedge dy_{i_k}\right) = \sum_I (\omega_I \circ F)\left(d(y_{i_1}\circ F) \wedge \dots \wedge d(y_{i_k} \circ F)\right)$$

d) If $M \subset \mathbb{R}^m$, $N \subset \mathbb{R}^n$ then $F^*(d\omega) = d(F^*\omega)$, with

$$d(f \, dx_1 \wedge \dots \wedge dx_m) = (df) \wedge dx_1 \wedge \dots \wedge dx_n$$

e) $(F \circ G)^* = G^* \circ F^*$ for $F: M \to N$, $G: P \to M$.

PROOF: Calculational, routine. (exercises) □

The exterior derivative on forms

This will extend the differential $d: C^\infty(M) \to \Omega^1(M)$

$$f \mapsto df$$

to a map $d: \Omega^k(M) \to \Omega^{k+1}(M)$.

Theorem. Let M be a smooth manifold with or without boundary. There are maps $d: \Omega^k(M) \to \Omega^{k+1}(M)$ and are uniquely determined by the following properties:

1) d is linear over $\mathbb{R}$

2) $\forall\ \omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$: $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^l (\omega \wedge (d\eta))$

3) $d^2 = 0$

4) $\forall f \in C^\infty(M)$, $\forall X \in \mathfrak{X}(M)$: $(df)(X) = Xf$ (as we defined before).

Pf: In a chart (U, φ) with coordinates (x_i) define

$$d\left(\sum_J \omega_J \cdot dx_{j_1} \wedge \dots \wedge dx_{j_k}\right) := \sum_J (d\omega) \wedge dx_{j_1} \wedge \dots \wedge dx_{j_k},$$

where the sum is taken over increasing multiindices $J = (j_1, \dots, j_k)$, $j_1 < \dots < j_k$.

Slightly modified way of saying this: $d\omega = \varphi^* d\left((\varphi^{-1})^* \omega\right)$

Let (V, ψ) be another chart, $U \cap V \neq \emptyset$. $\Rightarrow \varphi \circ \psi^{-1}$ is a diffeomorphism between open sets in $\mathbb{R}^n$.

$(\varphi \circ \psi^{-1})^* d\left((\varphi^{-1})^* \omega\right) = d\,(\varphi \circ \psi^{-1})^* (\varphi^{-1})^* \omega = d\,(\psi^{-1})^* \omega$ by the properties of the pullback.

Thus $\psi^* d(\psi^{-1})^* \omega = \psi^* (\varphi \circ \psi^{-1})^* d(\varphi^{-1})^* \omega =$

$= \varphi^* d(\varphi^{-1})^* \omega$.

Uniqueness First suppose $\omega_1 = \omega_2$ on an open set U.

Set $\eta = \omega_1 - \omega_2$ and $\Psi \in C^\infty(M)$ a bump function, $\Psi \equiv 1$ on $V \subset U$ open, $\operatorname{supp} \Psi \subset U$.

Then $\Psi \cdot \eta = 0$ and $d(\Psi\eta) = \Psi \cdot d\eta + d(\Psi) \wedge \eta$ and $\Psi(V) = 1$, $d\Psi|_V = 0$

$\Rightarrow$ $d\omega_1 = d\omega_2$ if. d satisfies 1) — 4).

Now let $\omega \in \Omega^k(M)$ and (U, φ), coordinates (x_i).

Then $\omega = \sum \omega_I \, dx_{i_1} \wedge \dots \wedge dx_{i_k}$ on this chart.

Choosing bump functions we can define $\tilde{\omega}_I$ as functions on $M \to \mathbb{R}$ which are smooth and get $\tilde{\omega} \equiv \omega$ for $\overset{\text{some}}{V} \subset U$ open

Then $d\omega = d\tilde{\omega} = d\left(\sum \tilde{\omega}_I \, dx_{i_1} \wedge \dots \wedge dx_{i_n}\right) = \sum d\tilde{\omega}_I \wedge dx_{i_1} \wedge \dots \wedge dx_{i_n}$

and this coincides with the differential defined before. □

Prop. Let $\omega \in \Omega^1(M)$, $X, Y \in \mathfrak{X}(M)$. Then $\underline{d\omega(X,Y)} = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \in C^\infty(M)$

Pf: Locally ω is a sum of terms $u\,dv$ where $u, v \in C^\infty(M)$.

We may only consider $\omega = u\,dv$ by linearity and locality.

$$d(u\,dv) = (du \wedge dv) \text{ and}$$

$$(du \wedge dv)(X,Y) = (du)(X) \cdot d(v)(Y) - (dv)(X)\,du(Y) =$$
$$= \underline{(Xu)(Yv) - (Xv)(Yu)}.$$

On the other hand,

$$X(u\,dv(Y)) - Y(u\,dv(X)) - u\,\underline{dv}([X,Y]) =$$
$$= (Xu)(Yv) + u(XYv) - (Yu)(Xv) + u(YXv) - u(XYv - YXv) =$$
$$= \underline{(Xu)(Yv) - (Xv)(Yu)}$$

□

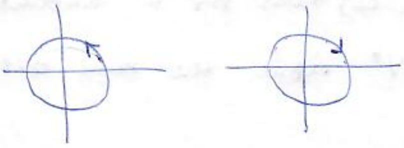
In general: for $\omega \in \Omega^k(M)$, $X_1, \dots, X_{k+1} \in \mathfrak{X}(M)$:

$$d\omega(X_1, \dots, X_{k+1}) = \sum_{1 \le i \le k+1} (-1)^{i-1} X_i \left(\omega(X_1, \dots, \hat{X}_i, \dots, X_k)\right) +$$
$$+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}).$$

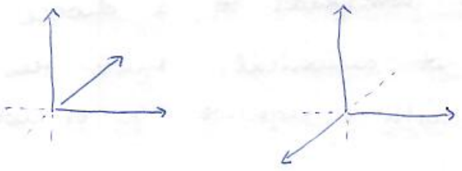
This is coordinate-independent but not very nice to compute.

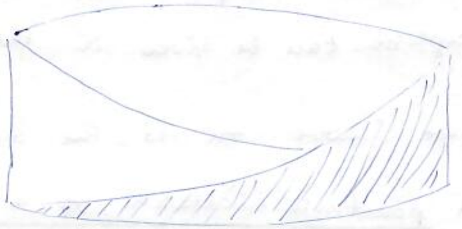
Orientation of manifolds

Orientation of 2-dim v.sp: choice between "clockwise" and "counterclockwise".



3-dim v.space: choice of left- / right-handedness





On the Möbius band,  goes to  by taking a turn ⇒ this is a non-orientable manifold.

Def. V n-dim v.space, $e = (e_1, \dots, e_n)$, $E = (E_1, \dots, E_n)$ are two bases for V. We say that e and E are consistently oriented if the transition matrix (B_i^j) which is defined by $e_i = \sum_j B_i^j E_j$ has positive determinant (we know that it is non-zero).

This defines an equivalence relation on the set of ordered bases of V, there are exactly 2 equivalence classes. (PO, simple linear algebra)

Notation. The eq. class of e is $[e]$.

Given an orientation given by e we say that E is positively oriented if $[e] = [E]$, otherwise E is negatively oriented.

Prop. Let V be a vector space of dimension $n \geq 1$. Then any $\omega \in \Lambda^n V^* \setminus \{0\}$ determines an orientation $\mathcal{O}_\omega$ on V as follows:

$(e_1, \dots, e_n)$ is positive $\Leftrightarrow \omega(e_1, \dots, e_n) > 0$

negative $\Leftrightarrow \omega(e_1, \dots, e_n) < 0$

(As $\omega \neq 0$ and e is a basis, $\omega(e) \neq 0$).

Two elements $\omega, \eta \in \Lambda^n V^* \setminus \{0\}$ determine the same orientation $\Leftrightarrow \omega = \lambda . \eta$ for $\lambda > 0$.

Pf: NTS: for $e = (e_1, \dots, e_n)$ with $\omega(e) > 0$ then $[e] = \{(E_1, \dots, E_n) = E \mid \omega(E) > 0\}$

Define $B: V \longrightarrow V$ by setting $BE_j := e_j$, this B is an invertible operator,

$$BE_j = e_j = \sum_j B_i^j E_j$$

Then $\underbrace{\omega(e_1,\dots,e_n)}_{>0} = \omega(BE_1,\dots,BE_n) = \underbrace{\det B}_{>0} \cdot \omega(E_1,\dots,E_n)$

$\Rightarrow \omega(E) > 0 \iff [E] = [e].$

□

Example. $(e_1,\dots,e_n)$ basis for V $\Rightarrow$ $(\varepsilon_1,\dots,\varepsilon_n)$ basis for V^* associated to $(e_1,\dots,e_n)$

Then $(e_1,\dots,e_n)$ and $\omega = \varepsilon_1 \wedge \dots \wedge \varepsilon_n \in \Lambda^n V^*$ define the same orientation.

Remark. $\Lambda^n V^* \cong \mathbb{R}$ and $\Lambda^n V^* \setminus \{0\} \cong \mathbb{R} \setminus \{0\}$.

This shows us that the choice of orientation corresponds to a choice of a connected component of $\mathbb{R}\setminus\{0\}$. This is not canonical, since the isomorphism can be given in two ways, either with a negative sign or not.

Now we move on to the orientation of manifolds.

Def. A pointwise orientation of a manifold M is a choice of orientation on T_pM for each $p \in M$.

(This is something we can always do since no requirement is given for these orientations.)

wrt a chosen pointwise orientation

Def. A local frame $(X_1,\dots,X_n)$ over $U \in M$ is positively oriented if $(X_1|_p,\dots,X_n|_p)$ is positively oriented. $\forall p \in U$.

Now we introduce a more restricted notion.

Def. A pointwise orientation of M is continuous if $\forall p \in M$ $\exists U$ nbh. of p for which there is a positively oriented local frame.

By an orientation of M we mean exactly this: a continuous pointwise orientation.

Formally, an oriented manifold is a pair $(M, \mathcal{O})$ where $\mathcal{O}$ is a continuous poitwise orient.

Def. An n-form $\omega \in \Omega^n(M)$ is non-vanishing if for any local frame $(X_1,\dots,X_n)$ with dual coframe $(dx_1,\dots,dx_n)$ we have that ω has the form

$$\omega = f \cdot dx_1 \wedge \dots \wedge dx_n \text{ where } f(x) \neq 0 \text{ on } U \text{ (} f \text{ is automatically smooth).}$$

Prop. Let M be a smooth manifold. (w/ or w/o boundary). Any non-vanishing n-form $\omega \in \Omega^n(M)$ (if it exists) determines a unique orientation on M. Conversely, if M is oriented, then there exists a non-vanishing $\omega \in \Omega^n(M)$ defining this orientation.

PROOF: ω defines a pointwise orientation by restriction to ordered bases of each T_pM.

NTS: this is continuous.

Given any local frame $(X_1, \dots, X_n)$ around $p \in U \subseteq M$ we have

$$\omega(X_1, \dots, X_n) = f \cdot dX_1 \wedge \dots \wedge dX_n$$

where f is non-vanishing.

Thus the orientation is indeed continuous (this was just a reformulation of the definition).

A bit more detailed explanation: the restriction to $V \subset U$ connected sets has $\operatorname{sgn} f|_V = \text{constant}$.

For the converse, suppose that M is oriented. Choose a cover $\{U_i\}$ by coordinate charts. We set $\omega_i := dx_1 \wedge \dots \wedge dx_n$, and choose a partition of unity χ_i subordinate to $\{U_i\}$, and set $\omega := \sum \chi_i \omega_i$.

Then on a local frame $(X^i_1, \dots, X^i_n)$ over U_i:

$$\omega(X^i_1, \dots, X^i_n) = \sum_j \chi_j \, dx^j_1 \wedge \dots \wedge dx^j_n \, (X^i_1, \dots, X^i_n)$$

basis trans. pointwise

$$= \sum_j \det(B^j_i) \, dx^i_1 \wedge \dots \wedge dx^i_n \, (X^i_1, \dots, X^i_n)$$

$$= \sum_j \chi_j(x) \det(B^j_i(x)) > 0$$

because M is oriented, so $\det(B^j_i(x)) > 0 \quad \forall x$.

□

Def. Let M be an oriented manifold. A chart (U, φ) is positively oriented if the frame $\left(\frac{\partial}{\partial x_i}\right)_i$ is positively oriented (here we mean the frame associated to this chart (U, φ)) for all $p \in U$.

An atlas $\{(U_i, \varphi_i)\}$ is pos. oriented if the transition maps

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$$

have positive Jacobian at all points.

Proposition. (PO) Let M be a smooth manifold. (w/ or w/o boundary) with $\dim M > 0$ and $\{(U_i, \varphi_i)\}$ a consistently oriented atlas. (i.e. all Jacobians of the transition maps are positive*).

Then there exists a unique orientation on M for which each (U_i, φ_i) is positively oriented.

Conversely, if M is oriented and $\partial M = \emptyset$ or $\dim M > 1$ then the collection of all positively oriented smooth charts is consistently oriented.

Remark.

"$\Leftarrow$" If $\dim M = 1$ and $\partial M \neq \emptyset$ then we require the boundary charts to satisfy that $\varphi(x) > 0$. This means that we cannot replace φ by $-\varphi$ to overcome non-consistent orientations.

This is why the restriction is needed in the statement.

If $\dim M > 1$, we can replace one coordinate with its negative, but that cannot be done in dimension 1 since the coordinate is required to be positive.

Def. Let $F: M \to N$ be a local diffeomorphism.

We say that F is orientation-preserving if $(dF)_p : T_pM \to T_pN$ maps pos. oriented bases to pos. oriented bases.

F is orientation-reversing if it maps pos. ori. bases to neg. ori. bases.

Prop. Let $F: M \to N$ be a local diffeomorphism and N be oriented. Then there is a unique orientation on M making F ~~positively oriented.~~ orientation-preserving.

PROOF: Let $\omega \in \Omega^n(N)$ be an orientation form. for N

Then the pullback $F^*\omega$ is an ori.-form for M.

Let (X_i) be a frame at $p \in U \subseteq M$. Then

$$F^*\omega\,(X_1, \dots, X_n) = \omega\big(dF(X_1), \dots, dF(X_n)\big) \neq 0$$

because $dF(X_i)$ is a local frame at $F(p) \in N$. since F is a local diffeomorphism.

Global Analysis 1, lecture 7–8

Boundary orientations

Does orientation on the ball give an orientation on the sphere?

Def. Let M be a manifold. An embedded submanifold is a subset $S \subseteq M$ that is itself a manifold without boundary in the subspace topology, endowed with smooth structure such that the inclusion map is a smooth embedding, i.e. $i : S \longrightarrow M$ is a homeomorphism onto its image and its derivative $di\big|_p : T_pS \longrightarrow T_pM$ is injective.

For a manifold with boundary we wish to show that the boundary is an embedded submanifold. Using this, we will induce an orientation on the boundary in a natural way.

Theorem. Let M be a smooth manifold with boundary. Let $p \in M$ s.t. for some chart (U, φ) with $\varphi(U) \subset \overline{\mathbb{H}}^n = \{(x_1, \dots, x_n) \mid x_n \geq 0\}$ we have that $\varphi(p) \in \partial\overline{\mathbb{H}}^n = \{(x_1, \dots, x_{n-1}, 0)\}$.

Then $\varphi(p) \in \partial\overline{\mathbb{H}}^n$ for any other chart (V, ψ) with $p \in V$ as well.

Corollary: $M = \operatorname{int} M \sqcup \partial M$ as a set-disjoint union.

PROOF OF THM: ↯ Suppose (V, ψ) is a chart with $\psi(V) \subset \mathbb{R}^n$ open and $\psi(p) \in \psi(V)$. 8.11.2017

Consider $J : \varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$.

By smooth compatibility, both J and J^{-1} admit smooth extensions to open sets containing $\psi(U \cap V)$ resp. $\varphi(U \cap V)$.

We choose a set $W \supset \varphi(U \cap V)$ open and admits a smooth extension of J^{-1}.

$x_0 := \psi(p)$, $y_0 := \varphi(p) = J(x_0)$ and $\eta : W \to \mathbb{R}^n$ for the extension of J^{-1}.

Choose an open ball B containing x_0 and lying s.t. $B \subset J^{-1}(W)$.

Then $(\eta \circ J)\big|_B = (J^{-1} \circ J)\big|_B = \mathrm{id}_B$. Hence the total (we are in $\mathbb{R}^n$) derivatives $D\eta$ and DJ satisfy $(D\eta)(J(x))(DJ)(x) = \mathrm{id}$ by the ordinary chain rule.

$\Rightarrow$ $(DJ)(x)$ is an invertible square matrix. $\forall x \in B$. Open mapping theorem:

$\Rightarrow$ $J : B \to \mathbb{R}^n$ is an open map $\Rightarrow$ $J(B)$ is an open set containing $y_0 = \varphi(p)$, contradicting the assumption that $\varphi(U) \subset \overline{\mathbb{H}}^n$ and $\varphi(p) \in \partial\overline{\mathbb{H}}^n$, ↯ as $J(B) \subset \varphi(U)$ □

The Open mapping theorem is not available for top. manifolds, and there the proof is much harder, but the thm. is still true.

Next goal: ∂M is a manifold.

Theorem. Let M be an n-dim manifold with boundary. Then ∂M is an $(n-1)$-dim manifold with charts as follows: for a boundary-chart (U, φ)

$$\varphi : \partial M \cap U \longrightarrow \mathbb{R}^n \xrightarrow{\pi_{n-1}} \mathbb{R}^{n-1} \quad \text{where } \pi_{n-1}(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}).$$

With these charts, this smooth structure $i : \partial M \to M$ is a smooth embedding.

Pf: ∂M is T2 and M2 in the relative topology.

(U, φ) a bdry-chart, $V := U \cap \partial M$ is open in the rel. topology.

$$\hat{V} := (\pi_{n-1} \circ \varphi)(V) \subset \mathbb{R}^{n-1}$$

$$\psi := \pi_{n-1} \circ \varphi : V \to \hat{V}$$

Then $\varphi(V) = \varphi(U) \cap (\mathbb{R}^{n-1} \times \{0\}) \Rightarrow \varphi(V)$ is open in $\mathbb{R}^{n-1}$

Moreover, ψ is a homeo ~~diffeomorphism~~* as $\psi^{-1} = \varphi^{-1} \circ i_0$ where $i_0 : \mathbb{R}^{n-1} \to \mathbb{R}^n$
$(x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_n)$

The only thing left to check is the smooth compatibility.

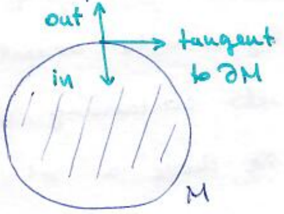
Suppose (V_1, ψ_1) and (V_2, ψ_2) are defined in the above way.

Then $\psi_2 \circ \psi_1^{-1} = \pi_{n-1} \circ \varphi_2 \circ \varphi_1^{-1} \circ i_0$ is a composition of smooth maps. ☐

* Then, when we checked that there is indeed a smooth structure, we can say that ψ is a diffeomorphism, but this word has no meaning before that.

Remark. ∂M is a manifold without boundary: $\partial\partial M = \emptyset$.

Let M be an oriented* manifold.
For $p \in \partial M$ the tangent T_pM decomposes as $T_pM = T_p\partial M \sqcup T_p^{in}M \sqcup T_p^{out}M$.



Where $$T_p^{in}M = \left\{ \sum_{k=1}^{n} v_k \cdot \frac{\partial}{\partial x_k} \;\middle|\; v_n > 0 \right\}$$

$$T_p^{out}M = \left\{ \sum_{k=1}^{n} v_k \cdot \frac{\partial}{\partial x_k} \;\middle|\; v_n < 0 \right\},$$

and $\left(\frac{\partial}{\partial x_k}\right)$ comes from some positively oriented chart. (Well-definedness comes from the positivity of the Jacobian.)

Lemma. M oriented w/ bdry ∂M. Then there is an $X \in \mathfrak{X}(M)$ s.t. $\forall p \in \partial M$: $X_p \in T_p^{\cancel{in}\,out}M$.

Pf: Choose a cover U_i of M by charts (U_i, φ_i) that are positively oriented and compatible for the orientation.

Then for each (U_i, φ_i) define $X_i := \frac{\partial}{\partial x_n}$.

Choose a partition of unity χ_i subordinate to U_i and set $X = \sum_{i \in A} X_i \chi_i$ ☐

* This notion can be generalised for non-oriented manifolds as well. Example: Möbius band.

Global Analysis, lecture 8

Prop. (Induced orientation on the boundary)

Let $n \geq 1$, M a smooth n-dimensional oriented manifold with boundary ∂M.

Then ∂M is orientable and the outward or the inward pointing vector fields define an orientation for ∂M.

Pf: Let $\omega \in \Omega^n(M)$ an orientation form and let N be an outward pointing vector field.

Define an $(n-1)$-form on ∂M by

$$i^*_{\partial M, N}\, \omega\, (X_1, \dots, X_{n-1}) := \omega(N, X_1, \dots, X_{n-1})$$

for a local frame (X_i) on ∂M.

Then

$$\omega_p\left(N_p, (X_1)_p, \dots, (X_{n-1})_p\right) \neq 0 \qquad \forall p \in M$$

because $N_p, (X_1)_p, \dots, (X_{n-1})_p$ is a basis for T_pM. This gives us the desired orientation form. □

Remark. This orientation on ∂M is sometimes called the Stokes orientation because of the role it plays in Stokes' theorem.

Example. The induced orientation on $\partial \bar{\mathbb{H}}^n$

Equip $\bar{\mathbb{H}}^n$ with the standard orientation inherited from $\mathbb{R}^n$: $dx_1 \wedge \dots \wedge dx_n$

Identify: $\partial \bar{\mathbb{H}}^n \longleftrightarrow \mathbb{R}^{n-1}$ via $(x_1, \dots, x_{n-1}, 0) \longleftrightarrow (x_1, \dots, x_{n-1})$,

and $\frac{-\partial}{\partial x_n}$ is the outward pointing vector field.

Then the standard coordinate frame for $\mathbb{R}^{n-1}$ is pos. oriented iff

$$\underbrace{\left[-\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\right]}_{}$$ gives the standard orientation.

$$= (-1)^n \left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_n}\right]$$ This means we pick up a minus sign when the dimension is odd.

Convention (but an important one!) to do this by putting the minus sign where we did.

Integration of differential forms

First we look at $\mathbb{R}^n$.

Def. A domain of integration $D \subset \mathbb{R}^n$. is a subset whose $\overset{\text{topological}}{}$ boundary $\bar{D} \setminus D^\circ$ has measure zero.

We need this notion to avoid weird space-filling curves and stuff like that.

By measure we mean the Lebesgue measure.

For such a domain, an n-form ω on $\bar{D}$ is given by

$$\omega = f\, dx_1 \wedge \ldots \wedge dx_n \quad \text{for } f : \bar{D} \to \mathbb{R} \text{ cont.}$$

Def. $\int_D \omega := \int_D f \, dx_1 \ldots dx_n$ „This is probably one of the more exciting definitions you have seen."

If $U \subset \mathbb{R}^n$ open and ω is compactly supported in U, then define

$$\int_U \omega := \int_D \omega \quad \text{where } D \text{ is any domain s.t. } \operatorname{supp} \omega \subset D \subset \bar{D} \subset U.$$

Clearly this does not depend on our choice for D.

Lemma. Let $U \subset \mathbb{R}^n$ or $U \subset \bar{\mathbb{H}}^n$, and ω comp. supported in U.

Then there exists a domain D s.t. $\underbrace{\operatorname{supp} \omega}_{K} \subset D \subset \bar{D} \subset U$.

Pf: $K := \operatorname{supp} \omega$ compact.

Cover K by finitely many open balls B_i s.t. $\bar{B_i} \subset U$. Then we have a finite union:

$$K = \bigcup_i B_i =: D$$

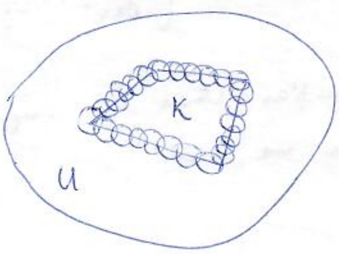
- As ∂B_i is a sphere in $\mathbb{R}^n$, these have Lebesgue measure 0.

$\Rightarrow$ ∂D has Lebesgue measure 0.

- In case of $\bar{\mathbb{H}}^n$: $\partial B_i \cap \partial \bar{\mathbb{H}}^n =$  $\cup$ , both terms have measure 0.

□

Note that ω does not play any role, this is an abstract statement about a compact set K.



Prop. D, E domains of integration in $\mathbb{R}^n$ or $\bar{\mathbb{H}}^n$,

G a smooth map $G : \bar{D} \to \bar{E}$ s.t.

$G : D \to E$ is an orientation-preserving diffeomorphism.

Then if ω is an n-form on $\bar{E}$, we have $\int_D G^* \omega = \int_E \omega$.

If G is ori-reversing: $\int_D G^* \omega = - \int_E \omega$.

Global Analysis I, lecture 8-9

PROOF: $(y_1, \dots, y_n)$ coordinates on E
$(x_1, \dots, x_n)$ — " — on D

Assume G preserves orientations.

$$\int_E \omega = \int f \, dy_1 \wedge \dots \wedge dy_n = \int f \, dy_1 \dots dy_n$$

$$= \int_D (f \circ G) \, \underbrace{\left| \det DG \right|}_{>0 \text{ since } G \text{ preserves orientations}} \, dx_1 \dots dx_n$$

$$= \int_D (f \circ G) \, (\det DG) \, dx_1 \dots dx_n = \int_D G^* \omega$$

If G reverses orientations: do the same, a minus sign appears when we remove the absolute value bars. □

Prop. Let U, V be open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$ and $G: U \to V$ an ori-preserving diffeomorphism, ω a compactly supported n-form on V.
Then $\int_V \omega = \int_U G^* \omega$.

PROOF: Take some open domain E s.t. $\operatorname{supp} \omega \subset E \subset \bar{E} \subset V$. Take $D := G^{-1}(E)$.

As G is a diffeomorphism, both domains satisfy the requirements to integrate, boundaries go to boundaries, measure-0 sets to measure-0 sets, interiors to interiors, $\operatorname{supp} G^* \omega \subset D$.

Hence the previous proposition proves our statement. □

This is the local knowledge we need to do integration on manifolds.

Integration on manifolds

Def.
Suppose $\omega \in \Omega^n(M)$ with compact support and $\operatorname{supp} \omega \subset U$ with U a chart (U, φ). Then set $\int_M \omega = \pm \int_{\varphi(U)} (\varphi^{-1})^* \omega$ where $\pm$ depends on whether (U, φ) is positively $(+)$ or negatively $(-)$ oriented. (This a priori uses that M is oriented.)

In general, cover M by charts (U_i, φ_i) and choose a partition of unity χ_i subordinate to $\{U_i\}$. Then define

$$\int_M \omega = \sum_i \left(\pm \int_{\varphi(U_i)} (\varphi_i^{-1})^* (\chi_i \omega) \right) = \sum_i \int_M \chi_i \omega$$

NTS: this is well-defined, independent of choices.

14. 11. 2017

M orientable manifold $\omega \in \Omega^n(M)$. compactly supported.

Any form gives a way to assign to a generalised paralelepiped, this will give us a signed volume.

Suppose ω is supported in a single chart (U, φ), i.e. $\operatorname{supp} \omega \subset U$.

$$\int_M \omega := \pm \int_{\varphi(U)} (\varphi^{-1})^* \omega.$$

this was our definition last week.
where $\varphi : U \to \mathbb{R}^n$, $\pm$ depends on the orientation of (U, φ).

Since $\operatorname{supp} \omega \subset M$ is cpt., $\operatorname{supp} (\varphi^{-1})^* \omega \subset \mathbb{R}^n$ is cpt. as well.

Prop. Suppose (U, φ) and (V, ψ) are charts, ω comp. supported n-form,

and $\operatorname{supp} \omega \subset U \cap V$.

Then $\int_{\varphi(U)} (\varphi^{-1})^* \omega = \pm \int_{\psi(V)} (\psi^{-1})^* \omega$, and thus $\int_M \omega$ is well-defined.

+ if (U, φ) and (V, ψ) are oriented in the same way,
− otherwise.

Pf: $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is an

- ori-preserving map if (U, φ) and (V, ψ) have the same orientation,
- ori-reversing map if (U, φ) and (V, ψ) have the opposite orientation.

Then:

$$\int_{\psi(V)} (\psi^{-1})^* \omega = \int_{\psi(U \cap V)} (\psi^{-1})^* \omega = \int_{\varphi(U \cap V)} \Big(\underbrace{(\psi \circ \varphi^{-1})}_{\text{diffeomorphism}}{}^{-1} \Big)^* (\psi^{-1})^* \omega =$$

$$= \pm \int_{\varphi(U \cap V)} (\varphi^{-1})^* \omega$$

+ if $\psi \circ \varphi^{-1}$ is ori-preserving,
− if $\psi \circ \varphi^{-1}$ is ori-reversing □

Now the inevitable happens: we cover M and use a partition of unity.

Def. M an oriented manifold of dimension n, $\omega \in \Omega^n(M)$ a compactly supported n-form. Let (U_i, φ_i) be any cover of M by charts, and χ_i an associated partition of unity.

Then define $\int_M \omega = \sum_i \int_M \chi_i \omega$ (note that $\operatorname{supp} \chi_i \omega \subset U_i$).

(There were a lot of choices, so a priori it is by no means clear that we have a well-defined integral here.)

Props. This definition of $\int_M \omega$ is independent of the choice of the charts (U_i, φ_i) and the partition of unity χ_i.

Global Analysis I, lecture 9

PROOF: (V_j, φ_j) another cover by charts,

η_j partition of unity.

For a fixed index i, we have that

$$\int_M \chi_i \omega = \int_M \underset{\substack{\uparrow \\ \text{finite sum}}}{\sum_j} \eta_j \chi_i \omega = \sum_j \int_M \eta_j \chi_i \omega$$

$$\Rightarrow \int_M \omega = \sum_i \int_M \chi_i \omega = \sum_i \sum_j \int \eta_j \chi_i \omega$$

By reversing the roles of η_j and χ_i, we get that

$$\int_M \omega = \sum_j \int_M \eta_j \omega = \sum_i \sum_j \int \eta_j \chi_i \omega.$$

□

Properties of the integral

Let M, N be oriented manifolds, both of dimension n.

Let $\omega, \eta \in \Omega^n(M)$ compactly supported. n-forms on M.

1) $\forall a, b \in \mathbb{R}: \int_M a\omega + b\eta = a \int_M \omega + b \int_M \eta$ PO.

2) If $-M$ denotes M with the opposite orientation, then

$$\int_{-M} \omega = - \int_M \omega$$

PO.

3) If ω is an positively oriented orientation form, then

$$\int_M \omega > 0.$$

This follows because in charts $\omega = f \, dx_1 \wedge \dots \wedge dx_n$ with $f > 0$,

And $\int_M \omega \geq \int_{(U,\varphi)} \omega > 0$.

4) If $F: M \to N$ is a diffeomorphism preserving orientation, then

$$\int_M \omega = \int_N F^* \omega$$

(Exercise)

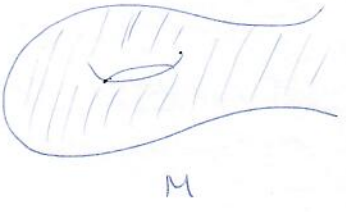
Stokes' Theorem. Let M be a manifold with boundary, $\dim M = n$, $\dim \partial M = n-1$.

Let $\omega \in \Omega^{n-1}(M)$ be compactly supported. Then we set

$$\int_{\partial M} \omega = \int_{\partial M} i^* \omega \qquad i : \partial M \hookrightarrow M \text{ inclusion} \quad \text{(i.e. restrict } \omega \text{ to the bdry).}$$

Note that $d\omega \in \Omega^n(M)$ and it is also compactly supported.

In this situation, we have that $\displaystyle\int_{\partial M} \omega = \int_M d\omega$.



If $\partial M = \emptyset$, the left hand side is understood to be 0.

PROOF: Suppose first that the thm. holds for forms ω supported in a single chart. (Then we will show that the general statement follows, and this will reduce the proof to the case of $\mathbb{R}^n$ and $\overline{\mathbb{H}}^n$.)

Choose (U_i, φ_i) and χ_i.

$$\sum_i \chi_i(x) = 1 \quad \forall x \in M$$

$$0 = d(1) = d \sum \chi_i = \sum d\chi_i$$

Then:

$$\int_{\partial M} \omega = \sum_i \int_{\partial M} \chi_i \omega \underset{\substack{\uparrow \\ \text{assumption}}}{=} \sum_i \int_M d(\chi_i \omega) =$$

$$= \underbrace{\sum_i \int_M d\chi_i \wedge \omega}_{0} + \sum_i \int_M \chi_i \, d\omega = \sum_i \int_M \chi_i \, d\omega = \int_M d\omega,$$

thus the general case indeed follows.

Hence it suffices to do the proof for $M = \mathbb{R}^n$ and $M = \overline{\mathbb{H}}^n$.

(Since $\partial \mathbb{R}^n = \emptyset$, we need to prove $\int_{\mathbb{R}^n} d\omega = 0$.)

Case 1. $M = \mathbb{R}^n$ $\Rightarrow$ $\exists R > 0 :$ $\operatorname{supp} \omega \subset [-R, R]^n$

Now $\omega = \sum_{i=1}^{n} \omega_i \, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$ for some ω_i.

Therefore
$$d\omega = \sum_{i=1}^{n} d\omega_i \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n.$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \omega_i}{\partial x_j} dx_j \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$$
$$= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x_i} dx_1 \wedge \ldots \wedge dx_n.$$

(This calculation also holds in $\overline{\mathbb{H}}^n$.)

$$\Rightarrow \int_M d\omega = \sum_{i=1}^{n} \int_{-R}^{R} \cdots \int_{-R}^{R} (-1)^{i-1} \frac{\partial \omega_i}{\partial x_i} dx_1 \wedge \ldots \wedge dx_n$$
$$= \sum_{i=1}^{n} \int_{-R}^{R} \cdots \int_{-R}^{R} (-1)^{i-1} \omega_i \Big|_{-R}^{R} dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$$
$$= 0 \quad \text{because } \omega_i(\ldots, \overset{i}{R}, \ldots) = \omega_i(\ldots, \overset{i}{-R}, \ldots) = 0. \quad ✓$$

Case 2. $M = \overline{\mathbb{H}}^n$. $\Rightarrow$ $\exists R > 0 :$ $\operatorname{supp} \omega \subset [-R, R]^{n-1} \times [0, R]$.

Then we have two integrals to compute: one on the boundary and one on $\overline{\mathbb{H}}^n$.

$$\int_{\partial \overline{\mathbb{H}}^n} \omega = \sum_{i=1}^{n} \int_{\partial \overline{\mathbb{H}}^n} \omega_i(x_1, \ldots, x_{n-1}, 0) \, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$$
$$= \int_{\partial \overline{\mathbb{H}}^n} \omega_n(x_1, \ldots, x_{n-1}, 0) \, dx_1 \wedge \ldots \wedge dx_{n-1} \quad \text{(every other term is 0)}$$
$$= (-1)^n \int_{-R}^{R} \cdots \int_{-R}^{R} \omega_n(x_1, \ldots, x_{n-1}, 0) \, dx_1 \wedge \ldots \wedge dx_{n-1}$$

$$\int_{\overline{\mathbb{H}}^n} \omega = \sum_{i=1}^{n} (-1)^n \int_{0}^{R} \int_{-R}^{R} \cdots \int_{-R}^{R} \frac{\partial \omega_i}{\partial x_i} dx_1 \wedge \ldots \wedge dx_n$$
$$= (-1)^n \int_{-R}^{R} \cdots \int_{-R}^{R} \omega_n(x_1, \ldots, x_{n-1}, 0) \, dx_1 \wedge \ldots \wedge dx_n \quad ✓$$

□

So we basically worked on a cube both times.

So far, we have no notion of distance, angle, volume etc.

To remedy this despicable situation, we introduce a new concept.

(Of course, if $M \subset \mathbb{R}^n$ for some n, we inherit such a notion, but such an embedding is not always available.)

Riemannian metrics

Motivation: need a product $\overset{\text{on } T_pM}{\text{that}}$ varies nicely when moving on M.

Def. Given a smooth manifold M, a Riemannian metric on M is a pairing

$$\begin{aligned} \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow C^\infty(M) \\ (X,Y) &\longmapsto g(X,Y) = \langle X, Y \rangle \end{aligned}$$

such that:

1) $g(X,Y) = g(Y,X)$

2) $g(X, f_1 Y_1 + f_2 Y_2) = f_1 \cdot g(X,Y_1) + f_2 \cdot g(X,Y_2) \qquad \forall f_1, f_2 \in C^\infty(M)$

(by 1), the same holds in the 1st variable)

3) $g(X,X)(p) > 0$ whenever $X(p) \neq 0$.

Lemma. Let M be a smooth manifold. Then there exists a Riemannian metric on M.

PROOF: $\{(U_i, \varphi_i)\}$ cover by coordinate charts.

Set $g_i(X, Y)(p) = \sum_{j=1}^{n} X_j(p) Y_j(p)$ where $X = \sum_j X_j \frac{\partial}{\partial x_j}$, $Y = \sum_j Y_j \frac{\partial}{\partial y_j}$,

i.e. g_i is the standard inner product.

Then choose χ_i partition of unity and set $g(X,Y) = \sum_i \chi_i g_i(X,Y)$.

Clearly $g_i(X,Y) \in C^\infty(M) \Rightarrow g(X,Y) \in C^\infty(M)$.

1) and 2) are trivially true, 3):
$$\begin{aligned} g(X,X)(p) &= \sum_i \chi_i(p) g_i(X,X)(p) \\ &= \sum_i \sum_j \chi_i(p) X_j^2(p) > 0 \end{aligned}$$

since at least one term is positive. □

This lemma tells us that the above definition is nothing special.

Example. (Euclidean metric on $\mathbb{R}^n$)

Define $g := \sum_{i=1}^{n} \underbrace{dx_i \cdot dx_i}_{\text{symmetric product of 1-forms}}$

This gives the usual inner product on each tangent space.

Global Analysis I, lecture 9–10

Example. (Round metric on S^n)

$$S^n = \{ (x_1, \dots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 = 1 \} \hookrightarrow \mathbb{R}^{n+1}$$ embedded submanifold.

$TS^n \hookrightarrow T\mathbb{R}^{n+1}$, $T_pS^n \hookrightarrow T_p\mathbb{R}^{n+1}$

Define g_{S^n} to be the restriction of the Euclidean metric to S^n.

Example. (Hyperbolic metric on $\mathbb{H}^n$)

$$\mathbb{H}^n = \{ (x_1, \dots, x_n) \mid x_n > 0 \}$$

Define $g = \sum_{i=1}^{n} \frac{dx_i \cdot dx_i}{x_n^2}$.

In H^2, we get $\frac{dx \cdot dx + dy \cdot dy}{y^2}$, the Poincaré metric.

15.11.2017

Notation. $\langle v, w \rangle_{g,p}$ the Riemannian metric in T_pM.

Metric ⟹ angles between tangent vectors, orthogonality, length.

Def. Let (M,g) be a Riemannian manifold, $p \in M$. A smooth orthonormal frame at p is a collection of $n = \dim M$ vector fields $(X_1, \dots, X_n)$ s.t. for some neighbourhood $U \ni p$ we have that $(X_1(q), \dots, X_n(q))$ is an orthonormal basis for T_qM w.r.t. $\langle \cdot, \cdot \rangle_g$ for all $q \in U$.

Warning: it is not true that for a chart (U, φ) the frame $\frac{\partial}{\partial x_i}$ is in general orthonormal. This is because the inner product has nothing to do with (U, φ) a priori.

Remember that $\frac{\partial}{\partial x_i}\Big|^{\varphi} = (\varphi^{-1})_* \frac{\partial}{\partial x_i}\Big|^{\mathbb{R}^n}$

Prop. $\forall p \in M$ $\exists U \ni p$ nbh. and a smooth orthonormal frame over U. Moreover, if M is oriented, then we can find an oriented orthonormal frame.

PROOF: Gram–Schmidt applied to the inner product g.

Let $(X_1, \dots, X_n)$ be a local frame, (e.g. $X_i = \frac{\partial}{\partial x_i}$ in some chart, but this choice is not necessary).

$E_1 := \frac{X_1}{|X_1|_g}$ where $|X_1|_g(p) = \langle X_1(p), X_1(p) \rangle_g^{1/2}$, $|X_1|_g \in C(M)$

($|X_1|_g \neq 0$ since we are talking about a frame.)

$$E_{k+1} := \frac{X_{k+1} - \sum_{i=1}^{k} \langle X_{k+1}, E_i \rangle_g E_i}{|X_{k+1} - \sum_{i=1}^{k} \langle X_{k+1}, E_i \rangle_g E_i|_g}$$

$E_1, \dots, E_k$ and $X_1, \dots, X_k$ have the same span, hence the denominator does not vanish at any $p \in M$.

→ $(E_1, \dots, E_n)$ orthonormal frame. If there is an orientation, replace E_1 by $-E_1$ if we got the wrong orientation. □

Prop. (M,g) a Riemannian manifold with or without boundary, oriented.

Then there is a unique orientation form ω_g with the following property:

$$\omega_g(E_1,\dots,E_n)=1 \qquad (*)$$

for any oriented local frame $(E_1,\dots,E_n)$. This is called the volume form ω_g

PROOF: Uniqueness: Let ω_g be a form satisfying $(*)$, $(E_1,\dots,E_n)$ an oriented local frame with dual coframe $(\varepsilon_1,\dots,\varepsilon_n)$.

Then $\omega_g = f\cdot \varepsilon_1\wedge\dots\wedge\varepsilon_n$ for some $f\in C^\infty(M)$.

By $(*)$, we have $f\equiv 1$. $\Rightarrow \omega_g = \varepsilon_1\wedge\dots\wedge\varepsilon_n$.

If $(E'_1,\dots,E'_n)$ is another frame, (oriented, local) with dual coframe $(\varepsilon'_1,\dots,\varepsilon'_n)$ then the frame transition matrix has determinant 1.

So $\varepsilon_1\wedge\dots\wedge\varepsilon_n = \varepsilon'_1\wedge\dots\wedge\varepsilon'_n$, therefore the local definition $\omega_g = \varepsilon_1\wedge\dots\wedge\varepsilon_n$ is well-defined. This shows the existence. □

Prop. (Volume form ω_g in a chart) (M,g) ori. R. manifold, $\dim M \geq 1$.

For any oriented chart with coordinates x_i we have

$$\omega_g = \sqrt{\det(g_{ij})}\; dx_1\wedge\dots\wedge dx_n$$

where $g_{ij} = g\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)$.

PROOF: In these coordinates we have $\omega_g = f\, dx^1\wedge\dots\wedge dx_n$ for some f.

To compute f, choose (E_i) oriented orthonormal frame with dual coframe (ε_i). Then write $\frac{\partial}{\partial x_i} = \sum_{j=1}^n A_i^j E_j$, so for f we have

$$f = \omega_g\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right) = \varepsilon_1\wedge\dots\wedge\varepsilon_n\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right) =$$

$$= \det(A_i^j)\cdot\varepsilon_1\wedge\dots\wedge\varepsilon_n(E_1,\dots,E_n) = \det(A_i^j).$$

On the other hand:

$$g_{ij} = g\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right) = g\left(\sum_h A_i^h E_h, \sum_l A_j^l E_l\right) \underset{\text{bilinearity}}{=}$$

$$= \sum_h\sum_l A_i^h A_j^l \underbrace{g(E_h,E_l)}_{\text{orthonormality: } \delta_{hl}} = \sum_h A_i^h A_j^h = (A^T A)_{i,j}$$

$$\Rightarrow \det(g_{ij}) = \det(A^TA) = (\det(A))^2 \Rightarrow f = \det(A) = \pm\sqrt{\det(g_{ij})}.$$

We find + if our chart is positively oriented. □

Global Analysis 1, lecture 10

(M,g) Riemannian, $\dim M = n$

$S \subseteq M$ submanifold embedded, $\dim S = k$

(S, g_S) restriction of g to TS. → S becomes a Riemannian manifold.

Example. $S^n \subset \mathbb{R}^{n+1}$

Now we can talk about vectors being orthogonal to S.

Def. We say that $v \in T_pM$ is normal to S of $\forall w \in T_pS \subset T_pM : \langle v, w \rangle_g = 0$.

$N_pS := \{ v \in T_pM \mid \forall w \in T_pS : \langle v, w \rangle_g = 0 \}$ is the normal space at p.

$NS := \bigsqcup_{p \in M} N_pS$ is the normal bundle of S inside M.

Note that $NS \subset TM$, and the restriction $TM \xrightarrow{\pi} M$ gives a projection $NS \xrightarrow{\pi_S} S$, and this makes NS a vector bundle of rank $n-k$.

In fact, $T_pM = T_pS \oplus N_pS$. If we didn't have a Riemannian metric, we could find such decompositions, but we wouldn't know how to choose them.

Prop. (M, g) Riemannian, $\dim M = n$, $S \subset M$ embedded submanifold. $\forall p \in S$ there is a local orthonormal frame $(E_1, \dots, E_k, E_{k+1}, \dots, E_n)$ such that $(E_1, \dots, E_k)$ restrict to an orthonormal frame on TS and $(E_{k+1}, \dots, E_n)$ restrict to an orthonormal frame for NS

PROOF: PO, follows from the decomposition $T_pM = T_pS \oplus N_pS$. Choose a frame in each direct summand. □

Integration of functions on (M, g)

Def. The volume integral of a function $f \in C^\infty(M)$ is defined to be

$$\underline{\int_M f \, dV_g} := \int_M f \cdot \omega_g$$

where ω_g is the volume form on (M, g).

Prop. Let f be a compactly supported function on a Riemannian manifold (M, g) and $f \geqslant 0$ then $\int_M f \, dV_g \geqslant 0$.

PROOF: The same statement for the integral of n-forms. □

The divergence theorem

Def. $X \in \mathfrak{X}(M)$ a vector field. $\underline{\beta(X)} \in \Omega^{n-1}(M)$ is defined by

$$\beta(X)(X_1, \dots, X_{n-1}) := \omega_g(X, X_1, \dots, X_{n-1}) \qquad \text{(using } g \text{ and the volume form determined by it)}$$

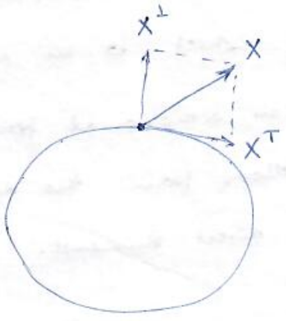
Also we have $\underline{\alpha} : C^\infty(M) \to \Omega^n(M)$
$f \mapsto f \cdot \omega_g$.

Lemma. Let (M, g) be an oriented Riemannian manifold, $S \subset M$ an embedded submanifold of codimension 1, i.e. $\dim S = \dim M - 1$

Let N be a normal vector field to S, i.e. $g(N, X) = 0 \quad \forall X \in \mathfrak{X}(S)$.
and $|N|_g = 1$

Then for any $X \in \mathfrak{X}(S)$ we have.

$$i_S^* \beta(X) = \langle X, N \rangle \omega_g = g(X, N) \omega_g.$$



where $i_S^* : S \hookrightarrow M$ is the embedding.

PROOF: Define $X^\perp := \langle X, N \rangle_g N$ orthogonal part
$X^T := X - X^\perp$ tangent part

Then $X = X^\perp + X^T$ and X^T is tangent to S,
$X^\perp$ is normal to S.

Moreover, $\beta(X) = \beta(X^T + X^\perp) = \beta(X^T) + \beta(X^\perp)$ since β is linear.

$$i_S^*\big(\beta(X^\perp)\big) \underset{\substack{\uparrow \\ \text{linearity}}}{=} \langle X, N \rangle_g \, i_S^*\big(\beta(N)\big) = \langle X, N \rangle_g \cdot \omega_g^S$$

where ω_g^S is the volume form of S corresponding to the induced metric.

(Note that S is oriented because N exists.)

Since X^T is tangent to S, we have for $E_1, \dots, E_{n-1}$ orthonormal frame that $\beta(X^T) = \omega_g(X^T, E_1, \dots, E_{n-1}) = 0$. for $X^T \in \operatorname{Span}\{E_1, \dots, E_{n-1}\}$

Thus $i_S^* \beta(X^T) = 0. \Rightarrow i_S^* \beta(X) = \langle X, N \rangle_g \cdot N$. □

Global Analysis I, lecture 10 - 11

Def. div : $\mathfrak{X}(M) \longrightarrow C^\infty(M)$ is defined by $\operatorname{div}(X) = \alpha^{-1}\, d\beta(X)$

(Note that α is invertible since M is orientable.)

Equivalently: $d\beta(X) = \operatorname{div}(X) \cdot \omega_g$

Theorem. (Divergence theorem) Let (M,g) be an oriented Riemannian manifold with boundary. For any compactly supported smooth vector field $X \in \mathfrak{X}(M)$ we have

$$\int_M \operatorname{div}(X)\, dV_g = \int_{\partial M} \langle X, N\rangle_g \, dV_g$$

where N is a unit vector field normal to ∂M.

PROOF: Stokes:
$$\int_M \operatorname{div}(X)\, dV_g = \int_M \operatorname{div}(X)\, \omega_g = \int_M d\beta(X) =$$
$$= \int_{\partial M} i_S^* \beta(X) \underset{\text{Lemma}}{=} \int_{\partial M} \langle X, N\rangle_g \, dV_g.$$

□

Let (M,g) be a Riemannian manifold, $S \subseteq M$ an embedded hypersurface, 21.11.2017
i.e. $\dim S = \dim M - 1$, and suppose that N is an unit normal vector field along S, that is,
$$g(N,N)(p) = 1 \qquad \forall p \in S$$
$$g(X,N)(p) = 0 \qquad \forall X \in \mathfrak{X}(S).$$

In this case, if M is oriented, then N determines an orientation on S.

In fact: $\omega_{\tilde{g}}^N (X_1, \dots, X_{n-1}) := \omega_g(N, X_1, \dots, X_{n-1})$

is a volume form on S, where ω_g is the volume form on M.

Here $\tilde{g}$ denotes the Riemannian metric g restricted to S.

Divergence theorem: $\displaystyle\int_M \operatorname{div}(X)\, dV_g = \int_{\partial M} \langle X, N\rangle_g \, dV_{\tilde{g}}.$

Here $X \in \mathfrak{X}(M)$ and N is an outward pointing normal vector field.

Then the orientation induced by N matches the Stokes orientation on ∂M.

The Div. Thm. holds in case ∂M is given the orientation induced by N.

Riemannian metric: $g: \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$

g is $\mathfrak{X}(M)$-bilinear, symmetric and positive definite.

For a vector space V with an inner product $\langle \cdot, \cdot \rangle$, there is an isomorphism $V \to V^*$ induced by $\langle \cdot, \cdot \rangle$.

$V^* = \{ \varphi : V \to \mathbb{R} \mid \varphi \text{ linear} \}$

Define $\varphi_v(w) := \langle v, w \rangle$, this gives $\varphi : V \to V^*$
$v \mapsto \varphi_v$

By finite dimensionality, φ is an isomorphism. (Special case of RRT.)

Similarly, a Riemannian metric g on M gives a map

$$\hat{g} : TM \to T^*M, \quad \hat{g}(v_p)(w_p) = \langle v_p, w_p \rangle_{g,p} \; (= g(v_p, w_p))$$

On vector fields $X, Y \in \mathfrak{X}(M)$ we have $\hat{g}(X)(Y) := g(X,Y)$. Then $\hat{g}(X) \in \Omega^1(M)$, and this gives an isomorphism $\hat{g} : \mathfrak{X}(M) \to \Omega^1(M)$

Injectivity of $\hat{g}$:

For if $\hat{g}(X) = 0 \in \Omega^1(M)$ then for all $Y \in \mathfrak{X}(M)$:

$0 = \hat{g}(X)(Y) = g(X,Y)$. Take $Y = X \Rightarrow g(X,X) = 0$, in particular $g(X,X)(p) = 0$ $\forall p \to X_p = 0 \; \forall p. \Rightarrow X = 0 \in \mathfrak{X}(M)$

Hence $\hat{g}$ is fiberwise injective, hence injective.

Surjectivity: both TM and T^*M are rank n vector bundles, hence $\hat{g}$ is fiberwise surjective $\Rightarrow$ surjective.

To summarise, a R. metric is an isomorphism between the tg and ctg bundles.

In coordinates: $g = \sum_{i,j} g_{i,j} \, dx_i \, dx_j$ where $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$

By def., $\hat{g}(X)(Y) = \sum_{ij} g_{ij} X_i Y_j$

Therefore $\hat{g}(X) = \sum_{ij} g_{ij} X_i \, dx_j$.

The matrix (g_{ij}) is invertible since it defines a (non-degenerate) pos. def. inner product.

The map $\hat{g}^{-1} : T^*M \to TM$ is given by

$$\hat{g}^{-1}(\omega) = \sum_{ij} (g^{-1})_{ij} \, \omega_j \frac{\partial}{\partial x_i}$$

Global Analysis 1, lecture 11

Def. For $f \in C^\infty(M)$ define grad $(f) \in \mathfrak{X}(M)$ by

$$\operatorname{grad}(f) := (\hat{g})^{-1}(df) \in \mathfrak{X}(M).$$

"It would be more convenient if the most common notation wasn't so poor in choice of letters." [about x and X]

Lemma. $\forall X \in \mathfrak{X}(M)\ \forall f \in C^\infty(M)$: $\langle \operatorname{grad} f, X\rangle_g = X \underset{\substack{\uparrow \\ \text{action}}}{\cdot} f$

PROOF:
$$\langle \operatorname{grad} f, X\rangle_g = g(\operatorname{grad}(f), X) = $$
$$= g\left(\hat{g}^{-1}(df), X\right) =$$
$$= \hat{g}\left(\hat{g}^{-1}(df)\right)(X) = df(X) = X \cdot f$$ □

Grad in coordinates:

$$\operatorname{grad} f = \sum_{ij} (g^{-1})_{i,j} \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j}$$

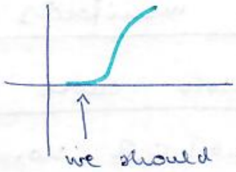
Therefore grad f is a smooth vector field since its coordinates are smooth as a vector field.

This recovers the usual gradient in $\mathbb{R}^n$ with the Euclidean metric.

Def. Let M be a manifold with boundary. A boundary defining function is a smooth map $f: M \to \mathbb{R}$ with

1) $f^{-1}(0) = \partial M$;

2) $df_p \neq 0 \quad \forall p \in \partial M$.



we should not reach zero like this; it should be pointy

Prop. M manifold w/ bdry. There is a bdry def. func.

PROOF: Cover M by charts (U_i, φ_i).

Define $f_i \in C^\infty(U_i)$ by

$$\begin{cases} f_i \equiv 1 & \text{if } U_i \text{ is an interior chart} \\ f_i(x_1, \dots, x_n) = x_n & \text{if } U_i \text{ is a bdry chart} \end{cases}$$

So $f_i(p) > 0$ if $p \in \operatorname{Int} M$ and $f_i(p) = 0$ if $p \in \partial M$.

$\exists \chi_i$ part. of unity, set $f := \sum_i \chi_i f_i$.

Then $f(p) > 0 \ \forall p \in \operatorname{Int}(M)$, $f(p) = 0 \ \forall p \in \partial M$. For $p \in \partial M$ and $v \in T_pM$:

$$df_p(v) = \sum_i \underbrace{f_i(p)\, d\chi_i|_p(v) +}_{\text{these sum up to 0, same as before}} \underbrace{\chi_i\, df_i|_p(v)}_{\neq 0 \text{, for } df_i|_p \neq 0 \text{ at bdry points}} \qquad \neq 0 \ \text{(as a form)}$$

⇑ ⇒ We can choose v for which $df_i|_p(v) > 0$ (and hence $df_j|_p(v) > 0 \ \forall j$, otherwise we get problems with the transition maps) □

Prop. Suppose (M, g) is a Riemannian manifold with boundary.

Then there is a unique outward pointing unit normal vector field N along ∂M.

PROOF: Uniqueness: suppose we have N.

Since $(T_p\partial M)^\perp \subset T_pM$ $(p \in \partial M)$ is 1-dimensional, there are only two choices for N_p. Outward-pointingness determines N_p uniquely.

Existence: let $f: M \to \mathbb{R}$ be a bdry def. function, and we set

$$N := \frac{-\operatorname{grad} f}{|\operatorname{grad} f|}.$$

grad $f \neq 0$ in a nbhd of the bdry since $df \neq 0$ there. Hence N is defined in a nbh. of the boundary.

Then check for $X \in \mathfrak{X}(\partial M)$:

$$\langle X, N\rangle (p) = \frac{-\langle \operatorname{grad} f, X\rangle}{|\operatorname{grad} f|} = \frac{-X_p f_p}{|\operatorname{grad} f|} = 0$$

because $f \equiv 0$.

And clearly $\langle N, N\rangle_g (p) = 1$ along ∂M. ☐

Corollary. The Möbius band is not a bdry of any R. manifold. ☐

Riemannian manifolds as metric spaces.

Def. A smooth curve segment in M is a smooth map $\gamma : [a,b] \to M$ where $[a,b] \subset \mathbb{R}$, closed interval.

A piecewise smooth curve is a continuous map $\gamma : [a,b] \to M$ s.t. $\exists\, a_0 = a < a_1 < \dots < a_n = b$ with $\gamma|_{[a_i, a_{i+1}]}$ smooth.

(Continuity is implied by the fact that all $\gamma|_{[a_i,a_{i+1}]}$ are smooth.)

Prop. If M is a connected smooth manifold then for every $p, q \in M$ there is a piecewise smooth curve $\gamma : [a,b] \to M$ connecting p and q, i.e. $\gamma(a) = p$, $\gamma(b) = q$.

PROOF: $p \in M$. Set $C_p = \{ q \in M \mid \exists \gamma : [a,b] \to M,$ piecewise smooth, $\gamma(a) = p, \gamma(b) = q\}$.

$p \in C_p$. We show that C_p is clopen, then connectedness gives $\emptyset \neq C_p = M$.

Let $q \in M$ and $U \subset M$, $q \in U$ a coordinate ball

Then any $q' \in U$ can be joined to q. Hence $q \in C_p \Rightarrow U \subset C_p$

$\Rightarrow C_p$ open. If $q \in \overline{C_p} \setminus C_p^\circ$ and $U \ni q$ a coord. ball.

$\Rightarrow U \cap C_p \neq 0$, $q' \in U \cap C_p$. q' can be connected to q in U

$\Rightarrow q \in C_p$, i.e. C_p is closed. ☐

Global Analysis I, lecture 11-12

Def. (Line integrals) Let $\gamma : [a,b] \to M$ be a piecewise smooth curve and $\omega \in \Omega^1(M)$ a differential 1-form.
Then the line in line integral of ω on γ is

$$\int_\gamma \omega := \int_{[a,b]} \gamma^* \omega = \int_a^b \gamma^* \omega$$

Note that $\gamma^* \omega \in \Omega^1(M)$ so we can integrate it.

Properties (follow from the respective properties of integrals of diff. forms)

1.) $\int_\gamma (c_1\omega_1 + c_2\omega_2) = c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2$

2.) If γ is constant then $\int_\gamma \omega = 0$.

3.) If $\gamma_1 = \gamma\big|_{[a,c]}$, $\gamma_2 = \gamma\big|_{[c,b]}$ then $\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega$ $(a \le c \le b)$

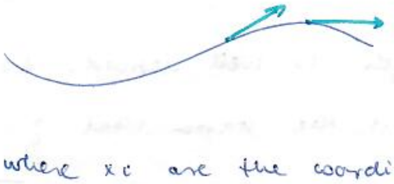
4.) If $F: M \to N$ is smooth then $\int_\gamma F^*\eta = \int_{F\circ\gamma} \eta$

Def. A reparametrisation of a piecewise smooth curve $\gamma : [a,b] \to M$ is another pw. s. c. $\tilde{\gamma} : [c,d] \to M$ s.t. $\exists\, \varphi : [c,d] \to [a,b]$ diffeomorphism for which $\tilde{\gamma} = \gamma \circ \varphi$.

We have $\int_\gamma \omega = \pm \int_{\tilde{\gamma}} \omega$, with "+" if φ is increasing,
"−" if φ is decreasing.
(This follows from the integration of diff. forms.)

Def. The tangent vector field to γ is defined to be the map

$$\gamma' : [a,b] \longrightarrow M$$
$$t \longmapsto (d\gamma)\left(\frac{d}{dx}\Big|_t\right)$$ where x is the coordinate on $[a,b]$



Then $\gamma'(t) = \sum_i \frac{d\gamma_i}{dx}(t) \frac{\partial}{\partial x_i}\Big|_{\gamma(t)} =$

$= \frac{d}{dx}\Big|_{x=t} \gamma(x).$ where x_i are the coordinates on M.

Def. The length of a pw. smooth curve in (M,g) is

$$L_g(\gamma) := \int_a^b |\gamma'(t)|_g \, dt = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t)\rangle_g} \, dt .$$

Then the distance function is

$$d_g(p,q) := \inf \{ L_g(\gamma) \mid \gamma \text{ pw. s. c. from } p \text{ to } q \} .$$

22.11.2017

Prop. The line integral $\int_\gamma \omega$ admits the expression

$$\int_\gamma \omega = \int_a^b \underbrace{\omega_{\gamma(t)}(\gamma'(t))}_{\text{function}} dt .$$

PROOF: In coordinates: $\omega = \sum_i \omega_i \, dx_i$, $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$

Then $\gamma^* \omega(t) = \sum_i (\omega_i \circ \gamma)(t) \, d\gamma_i(t) = \sum_i \omega_i(\gamma(t)) \cdot \gamma_i'(t) \, dt$

$$= \sum_i \omega_i(\gamma(t)) \cdot dx_i(\gamma'(t)) = \omega_{\gamma(t)}(\gamma'(t)) \, dt$$ □

Here we only did calculus in the chart, only the notation is new.

Prop. If $\gamma : [a,b] \to M$ and $f \in C^\infty(M)$, then $\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$

PROOF: Corollary of Stokes, but could be derived more elementarily. □

If (M,g) is Riemannian, then we can talk about length, as above.

Prop. Let (M,g) be a Riemanian manifold and $\gamma : [a,b] \to M$ a piecewise smooth curve. If $\tilde{\gamma} : [c,d] \to M$ is a reparametrisation, then $L_g(\gamma) = L_g(\tilde{\gamma})$.

PROOF: Let $\varphi : [c,d] \to [a,b]$ be s.t. $\tilde{\gamma} = \gamma \circ \varphi$. Assume $\varphi' > 0$ ($\varphi' < 0$ is similar)

$$L_g(\tilde{\gamma}) = \int_c^d |\tilde{\gamma}'(t)|_g \, dt = \int_c^d \left| \frac{d}{dx} (\gamma \circ \varphi)(t) \right|_g dt = \int_c^d |\varphi'(t) \cdot \gamma'(\varphi(t))|_g \, dt =$$

$$= \int_c^d |\gamma'(\varphi(t))|_g \cdot \varphi'(t) \, dt = \int_a^b |\gamma'(s)|_g \, ds = L_g(\gamma)$$ □

Hence the length is well-defined. As above, we introduce the notion of distance; this is the reason that g is called a R. metric.

If M is connected, then $d_g(p,q) < \infty$ since there is a path between them.

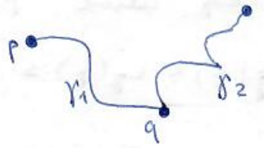
The disconnected case will be adressed later.

Global Analysis I, lecture 12

Clearly $d(p,p)=0$ and $d_g(p,q) = d_g(q,p)$. So to prove that d_g is a metric, all that's left to prove is the triangle inequality, and the fact that $d_g(p,q) > 0 \quad \forall p \neq q$.

<u>Triangle inequality</u>: γ_1 path $p \rightsquigarrow q$, γ_2 path $q \rightsquigarrow r$.

Then the concatenation γ of γ_1 and γ_2 is a path $p \rightsquigarrow r$. $d_g(p,r) \leq L_g(\gamma) = L_g(\gamma_1) + L_g(\gamma_2)$



inf on rhs $\rightarrow$ $d_g(p,r) \leq d_g(p,q) + d_g(q,r)$. ✓

<u>Lemma</u>. Let g be a Riemannian metric on an open set $U \subset \mathbb{R}^n$. Let $\bar{g}$ be the Euclidean metric, $\bar{g} = \sum_i (dx_i)^2$. Given a compact subset $K \subset U$ there exist $c, C \in \mathbb{R}_{>0}$ const. s.t. $\forall x \in K \ \forall v \in T_x\mathbb{R}^n$ the inequality

$$c\,|v|_{x,\bar{g}} \leq |v|_{x,g} \leq C \cdot |v|_{x,\bar{g}}$$

holds true. (c and C depend on K).

PROOF: Let $L_K := \{ (x,v) \in T_x\mathbb{R}^n \mid x \in K, \ |v|_{\bar{g}} = 1 \} \ (\simeq K \times S^{n-1})$

Identify $T\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$, and note that L_K is compact.

The norm map $(x,v) \mapsto |v|_{x,g}$ is continuous.

Therefore it is bounded, i.e. $\exists c, C$: $c \leq |v|_{x,g} \leq C$, the bound below comes from the fact that 0 is never achieved: $|v|_{(g,x)} \neq 0$.

$\lambda(x) := |v|_{\bar{g},x} \quad \forall v \in T_x\mathbb{R}^n$

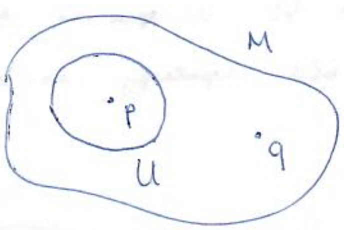
Then $c.\lambda \leq \lambda \cdot |\lambda^{-1} v|_{(g,x)} \leq \lambda . C$ because $\lambda^{-1} v \in L_K$.

$\Rightarrow c\,|v|_{\bar{g},x} \leq |v|_{(g,x)} \leq C . |v|_{(\bar{g},x)}$ □

<u>Theorem</u>. Let (M,g) be a Riemannian connected manifold. Then the Riemannian distance function is a metric on M, and the metric topology coincides with the given topology on M.

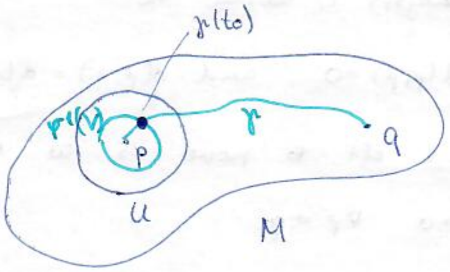
PROOF: NTS $d_g(p,q) > 0 \ \forall p \neq q$, all other properties have been already established.

Take a chart (U,φ) containing p but not q.



Let $\bar{g}$ be the Euclidean metric on $\varphi(U)$.

For any open ball $V \subset \varphi(U)$ centered at p we have



$$c\cdot|v|_{\bar{g}} \leq |v|_g \leq C\cdot|v|_{\bar{g}} \quad \forall (x,v)\in L\bar{V}$$

Then for any curve $\gamma : [a,b] \longrightarrow M$ it follows that

$$c\cdot L_{\bar{g}}(\gamma) \leq L_g(\gamma) \leq C\cdot L_{\bar{g}}(\gamma)$$

Now let $\gamma : [a,b] \longrightarrow M$ be a pw.s.c. from p to q. and set

$$t_0 := \inf\{t\in[a,b] \mid \gamma(t)\notin \bar{V}\}$$

$\Rightarrow$ $\gamma(t_0)\in\bar{V}\setminus V$ by continuity, $\gamma(t)\in V$ whenever $t_0 \geq t \geq a$

(In some places, there should be a φ or a φ^{-1}, but even in this imprecise way, it is clear what we are saying. In our mind, we identify V and $\varphi^{-1}(V)$, U and $\varphi(U)$, γ and $\varphi\circ\gamma$.)

Hence $L_g(\gamma) \geq L_g(\gamma|_{[a,t_0]}) \geq c\cdot L_{\bar{g}}(\gamma|_{[a,t_0]}) \geq c\cdot d_g(p,\gamma(t_0)) = c\cdot\varepsilon > 0$.

where V is an ε-ball. $\Rightarrow$ $d_g(p,q) = \inf\limits_{\gamma} L_g(\gamma) \geq c\varepsilon > 0$.

To see that d_g induces the manifold topology, let $U\subset M$. open. in the manifold topology. The above argument shows that $\forall p\in U$ $\exists$ coordinate ball V of radius ε around p, and this V contains the metric ball of radius $c\cdot\varepsilon$ around p.

$\Rightarrow$ U can be written as an union of metric balls $\Rightarrow$ U is open in the metric topology.

Let W be open in the metric topology, $p\in W$, and V a coordinate ball around p.

Let $\bar{g}$ denote the Eu. metric on V, denote the radius of V by r.

Then take $\varepsilon < r$ small enough s.t. the metric ball of radius $C.\varepsilon$ around p. is in W.

$V_\varepsilon := \{q\in\bar{V} \mid d_{\bar{g}}(p,q) < \varepsilon\}$. Then $d_g(p,q) \leq C.d_{\bar{g}}(p,q) < C.\varepsilon$.

$\Rightarrow$ V_ε is contained in the metric ball of radius $C.\varepsilon$, which is in W, and V_ε is open in M $\Rightarrow$ every point of W has an open nbh. in the manifold topology $\Rightarrow$ W is manifold-open.

□

Global Analysis 1, lecture 12-13

Corollary. Every smooth manifold M is metrisable.

PROOF: Assume M is connected, and choose a Riemannian metric g on M.

R. manifolds are metrisable $\Rightarrow$ M is metrisable.

Not connected case: metrise the connected components, and put the different components at a fixed finite distance.

Different approach for the non-conn. case: choose $x_i \in M_i$ $\forall i$ conn. component, and let $d(x,y) = d(x,x_i) + d(y,x_j) + 1$ when $x \in M_i \neq M_j \ni y$. □

Goal. To make sense of the notion of a straight line. (We intend to do geometry or physics.)

$\mathbb{R}^2$: line $\Leftrightarrow$ $\gamma''(t) = 0$.

This poses a problem in the general case since $\gamma'(t)$ is a vector field, so $T_pM \ni \gamma'(t)$. To "differentiate" this, we need to make sense of $\frac{\gamma'(t)-\gamma'(s)}{t-s}$, but $\gamma'(t)$ and $\gamma'(s)$ are in the distinct spaces T_pM and T_sM $\Rightarrow$ their difference does not make sense.

Tensor fields

28.11.2017

Def. V a vector space. A covariant k-tensor on V is an element $\xi \in \underbrace{V^* \otimes \dots \otimes V^*}_{k \text{ times}} = (V^*)^{\otimes k}$.

A contravariant k-tensor is a $\xi \in V \otimes \dots \otimes V = V^{\otimes k}$ (k-fold tensor product).

A covariant k-tensor can be viewed as a multilinear functional $\underbrace{V \times \dots \times V}_{k \text{ times}} \longrightarrow \mathbb{R}$ via $(\xi_1 \otimes \dots \otimes \xi_k)(v_1, \dots, v_k) = \xi_1(v) \cdot \ldots \cdot \xi_k(v_k)$

with $\xi_i \in V^*$, $v_i \in V$, and then extend by linearity.

Note that a general element $\xi \in (V^*)^{\otimes k}$ is of the form $\sum_{j=1}^{m} \xi_1^j \otimes \dots \otimes \xi_k^j$.

Conversely, a contravariant k-tensor defines a multilinear functional $\underbrace{V^* \times \dots \times V^*}_{k \text{ times}} \longrightarrow \mathbb{R}$ via $(v_1 \otimes \dots \otimes v_k)(\xi_1, \dots, \xi_k) = \xi_1(v_1) \cdot \ldots \cdot \xi_k(v_k)$.

Def.
A k-tensor α is symmetric if $\forall \sigma \in S_k$ we have

$$\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \alpha(v_1, \dots, v_k).$$

A k-tensor α is alternating if $\forall \sigma \in S_k$ we have

$$\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \mathrm{sgn}(\sigma) \cdot \alpha(v_1, \dots, v_k). \qquad \text{(Like diff. forms.)}$$

Def. Symmetrisation: $\mathrm{Sym}(\alpha)(v_1, \dots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$ average over the action of the symmetric group.

Antisymmetrisation: $A(\alpha)(v_1, \dots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$.

We don't have $\alpha = \mathrm{Sym}(\alpha) + A(\alpha)$. (Luckily, we don't even need it.)

If α, β are symmetric, then $\alpha \otimes \beta$ need not be symmetric.

We would like a product on symm. tensors like the wedge on alternating ones.

Def. Symmetric product of a k-tensor α and an l-tensor β:

$$\alpha \cdot \beta(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_k} \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \cdot \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}).$$

This is a symmetric $(k+l)$-tensor.

The symmetric product is equal to $\mathrm{Sym}(\alpha \otimes \beta)$.

The symmetric product is commutative and distributive:

$$\alpha\beta = \beta\alpha, \quad (a\alpha + b\beta)\cdot\gamma = a\alpha\cdot\gamma + b\beta\cdot\gamma. \qquad \forall a, b \in \mathbb{R}.$$

Def.
Let M be a manifold. The bundle of covariant k-tensors is

$$T^k M := (T^*M)^{\otimes k} = \coprod_{p \in M} (T_p^* M)^{\otimes k}.$$

The bundle of contravariant k-tensors is $T_k M = (TM)^{\otimes k} = \coprod_{p \in M} (T_p M)^{\otimes k}$

The bundle of mixed tensors of type (k, l) is

$$T^k_l(M) := (T^*M)^{\otimes k} \otimes (TM)^{\otimes l} = \coprod_{p \in M} (T^*_p M)^{\otimes k} \otimes (T_p M)^{\otimes l}.$$

Using the vector bundle chart lemma, we obtain a vector bundle structure on $T^k_l(M)$ by defining maps $\tau_{ij}^{(k,l)} : U_i \cap U_j \longrightarrow GL((\mathbb{R}^*)^{nk} \otimes \mathbb{R}^{nl})$,

$$\tau_{ij}^{(k,l)}(p)(w_1 \otimes \dots \otimes w_k, v_1 \otimes \dots \otimes v_l) = \tau_{ij}^{T^*M}(p) w_1 \otimes \dots \otimes \tau_{ij}^{T^*M}(p) w_k \otimes \tau_{ij}^{TM}(p) v_1 \otimes \dots \otimes \tau_{ij}^{TM}(p) v_l.$$

Global Analysis 1, lecture 13

Def. A tensor field of type (k,ℓ) is a (smooth) section of $T^k_\ell M$.

(One could also consider continuous sections, for example.)

Recall that for a Riemannian manifold (M,g) we have the duality map $\hat{g}: TM \xrightarrow{\sim} T^*M$.

This induces maps $T^k_\ell M \longrightarrow T^{k+1}_{\ell-1} M$ by applying $\hat{g}$ to one of the contravariant components.

$\hat{g}^{-1}$ induces maps $T^k_\ell M \longrightarrow T^{\ell-1}_{k+1} M$ by applying $\hat{g}^{-1}$ to one of the covariant arguments.

Def. Lastly, for a contravariant 2-tensor on a Riemannian manifold we define its trace by

$$\mathrm{Tr}_g : X_p \otimes Y_p \longmapsto g(X,Y)(p)$$

where $X_p, Y_p \in T_pM$

On 2-tensor fields this gives the map

$$\begin{tikzcd} \mathfrak{X}(M) \times \mathfrak{X}(M) \arrow[r, "g"'] \arrow[d] & C^\infty(M) \\ \mathfrak{X}(M) \otimes \mathfrak{X}(M) \arrow[ur] & \end{tikzcd}$$

„I don't know how this would popularise. I want to leave the physicists, of course. [...] This is an unfortunacy of history."

(About why covariant and contravariant are called this way.)

If $\gamma: [a,b] \longrightarrow M$ is a smooth curve in a Riemannian manifold (M,g) then the expression $\gamma''(t_0) = \lim\limits_{t\to t_0} \dfrac{\gamma'(t) - \gamma'(t_0)}{t-t_0}$ does not make sense

because $\gamma'(t)$ and $\gamma'(t_0)$ are in different vector spaces.

To make sense of $\gamma''(t_0)$, we introduce the notion of connection (i.e. connecting different tangent spaces). This can be done for arbitrary vector bundles, we won't do it in full generality.

Def. Let $\pi: E \to M$ be a smooth vector bundle over a manifold M, with space of smooth sections $\Gamma^\infty(E)$.

A connection in E is a linear map

$$\nabla: \Gamma^\infty(E) \longrightarrow \Gamma^\infty(E) \underset{C^\infty(M)}{\otimes} \Omega^1(M)$$

with the property that $\forall f \in C^\infty(M)$, $\forall Y \in \Gamma^\infty(E)$ it holds that

$$\nabla(Y \cdot f) = \nabla(Y) \cdot f + Y \otimes df \qquad \text{(derivation-like behaviour).}$$

Using the pairing
$$\begin{aligned} \mathfrak{X}(M) \times \Omega^1(M) &\longrightarrow C^\infty(M) \\ (X, \omega) &\longmapsto \omega(X) \end{aligned}$$

we obtain a pairing
$$\begin{aligned} \mathfrak{X}(M) \times \Gamma^\infty(E) \underset{C^\infty(M)}{\otimes} \Omega^1(M) &\longrightarrow \Gamma^\infty(E) \\ (X, Y \otimes \omega) &\longmapsto Y\omega(X). \end{aligned}$$

We write $(Y \otimes \omega)(X)$ for $Y\,\omega(X)$.

Def. Then given $\nabla: \Gamma^\infty(E) \to \Gamma^\infty(E) \otimes \Omega^1(M)$ and $X \in \mathfrak{X}(M)$, define

$$\nabla_X Y := \nabla(Y)(X)$$

For $f \in C^\infty(M)$, we have $\nabla_{X \cdot f} Y = f\, \nabla_X Y$.

Lemma. Let $\nabla: \Gamma^\infty(E) \to \Gamma^\infty(E) \underset{C^\infty(M)}{\otimes} \Omega^1(M)$ be a connection. Then $\nabla_X Y(p)$ only depends on the values of X, Y in a small nbh. of $p \in M$.

PROOF: First we show that if $Y = 0$ on a nbh. U of p, then $\nabla_X Y = 0$ on U as well.

Choose a bump function $\varphi \in C^\infty(M)$ with $\operatorname{supp} \varphi \subset U$ and $\varphi(p) = 1$.

Consider $\varphi Y \in \Gamma^\infty(E)$.

Then for any $X \in \mathfrak{X}(M)$ we have

$$\begin{aligned} \nabla_X(\varphi Y) &= (\varphi \nabla_\cdot Y + Y \otimes d\varphi)(X) \\ &= \varphi \cdot \nabla_X Y + Y(X\varphi) \end{aligned}$$

Also $\nabla_X(\varphi Y) = 0$ because $\varphi \cdot Y = 0$, since $\operatorname{supp} \varphi \subset U$ and $Y = 0$ on U.

$\Rightarrow \varphi \nabla_X Y + Y(X\varphi) = 0$.

$Y(X\varphi) = 0$ since $\operatorname{supp} \varphi \subset U$ and $Y = 0$ on U. $\Rightarrow \varphi \nabla_X Y = 0$ at p

$\varphi(p) = 1 \Rightarrow (\nabla_X Y)(p) = 0$.

Global Analysis 1, lecture 13 – 14

For X, the argument is similar but easier. (Check it.) In fact, for X it suffices to only know X_p.

Locally $X = \sum X_i \frac{\partial}{\partial x_i}$.

If $X_i(p) = 0$ $\forall i$, then $\forall Y \in \Gamma^\infty(E)$:

$$\nabla_X Y\ (p) = \left(\sum X_i \nabla_{\frac{\partial}{\partial x_i}} Y\right)(p) = \sum X_i(p) \nabla_{\frac{\partial}{\partial x_i}} Y = 0.$$

Thus we can write $\nabla_{X_p} Y$ for $\nabla_X Y(p)$. (Think of this as a directional derivative.)

Def. An affine or linear connection is a connection in $TM = E$.

Expressions with local frame:

Let (E_i) be a local frame for E at p, and write

$$\nabla_{E_i} E_j = \sum_k \Gamma_{ij}^k E_k$$

with Γ_{ij}^k defined in a nbhd of p.

Def. The functions Γ_{ij}^k are called the Christoffel symbols of ∇ w.r.t. (E_i).

Lemma. Let $\nabla : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M) \otimes \Omega^1(M)$ be a linear connection. For $X, Y \in \mathfrak{X}(M)$ and a local frame (E_k), writing

$$X = \sum_k X_k E_k, \qquad Y = \sum_j Y_j E_j$$

we have $\nabla_X Y = \sum_{i,j,k} \left(X(Y_k) + X_i Y_j \Gamma_{ij}^k\right) E_k$.

(This says that the Christoffel symbols tell us everything we need to know about ∇.)

PROOF: $\nabla_X Y = \sum_j \nabla_X (Y_j E_j) = \sum_{i,j} X(Y_j) E_j + Y_j \nabla_{X_i E_i} E_j =$

$$= \sum_{i,j} X(Y_j) E_j + X_i Y_j \nabla_{E_i} E_j = \sum_{i,j,k} X(Y_j) E_j + X_i Y_j \Gamma_{ij}^k E_k =$$

$$= \sum_{i,j,k} \left(X(Y_k) + X_i Y_j \Gamma_{i,j}^k\right) E_k$$

$\square$

29.11.2017

Lemma. For an open set $U \subset \mathbb{R}^n$, there is a one-to-one correspondence between connections in TU and the choice of n^3 functions $\Gamma_{ij}^k \in C^\infty(U)$ $1 \le i,j,k \le n$

via $$\nabla_X Y = \sum_{i,j,k} \left(X(Y_k) + X_i Y_j \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k}$$

PROOF: Any connection takes this form by previous discussion (we use the standard frame for E).

Easy to verify: if this is satisfied, ∇ is a connection. □

Prop. Every manifold admits a linear connection.

PROOF: Cover M with charts U_i.

By the Lemma, there is a connection ∇_i in each chart:

$$\nabla_i : \mathfrak{X}(U_i) \times \mathfrak{X}(U_i) \longrightarrow \mathfrak{X}(U_i)$$

Note that whatever Christoffel symbols we choose, we get a connection ∇_i.

Then take a part. of unity χ_i subordinate to U_i, and set

$$\nabla := \sum_i \chi_i \nabla_i$$

Leibniz rule:

$$\nabla_X (Yf) = \sum_i \chi_i \nabla_{i,X}(Yf) = \sum_i \left(\chi_i \nabla_{i,X}(Y) \cdot f + \chi_i Y \otimes df \right)$$
$$= \nabla_X(Y) \cdot f + Y(Xf).$$

(Note that we need $\sum \chi_i$ to be exactly 1.) □

A linear combination of connections ∇_1, ∇_2: $\nabla = \lambda_1 \nabla_1 + \lambda_2 \nabla_2$ $\lambda_1, \lambda_2 \in \mathbb{R}$ is a connection iff $\lambda_1 + \lambda_2 = 1$, calculation same as above.

I.e. only convex combinations of connections are connections.

Lemma. Let ∇ be a linear connection on M. There is a unique connection in each tensor bundle $T^k_\ell M$, also denoted by ∇, s.t. the following hold:

a) ∇ agrees with the given connection on TM.

b) On $T^0 M = M \times \mathbb{R}$ (so $\Gamma^\infty(T^0 M) = C^\infty(M)$), ∇ is given by

$\nabla(f) = df$, or $\nabla_X f = Xf$.

c) ∇ satisfies the following Leibniz rule for tensor products:

$$\nabla_X (F \otimes G) = \nabla_X(F) \otimes G + F \otimes \nabla_X(G) \qquad F \in \Gamma^\infty(T^k_\ell M),\ G \in \Gamma^\infty(T^{k'}_{\ell'} M)$$

d) Recall $$\mathrm{Tr} : \mathfrak{X}(M) \times \Omega^1(M) \longrightarrow C^\infty(M).$$
$$(X, \omega) \longmapsto \omega(X)$$

This gives a map $\mathrm{Tr} : T^k_\ell M \longrightarrow T^{k-1}_{\ell-1} M$, $k, \ell > 0$.

Then $\nabla_X(\mathrm{Tr}\, Y) = \mathrm{Tr}(\nabla_X Y)$ $\forall X \in \mathfrak{X}(M)$, $Y \in \Gamma^\infty(T^k_\ell M)$.

Global Analysis 1, lecture 14

Such a connection must also satisfy

i) $\forall \omega \in \Omega^1(M)$, $X, Y \in \mathfrak{X}(M)$: $\nabla_X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y)$

ii) $\forall F \in \Gamma^\infty(T^k_\ell M)$ and vector fields Y_i, 1-forms ω_j we have

$$(\nabla_X F)(\omega_1, \dots, \omega_\ell, Y_1, \dots, Y_k) = X(F(\omega_1, \dots, \omega_\ell, Y_1, \dots, Y_k)) -$$
$$- \sum_{j=1}^{\ell} F(\omega_1, \dots, \nabla_X \omega_j, \dots, \omega_\ell, Y_1, \dots, Y_k) - \sum_{i=1}^{k} F(\omega_1, \dots, \omega_\ell, Y_1, \dots, \nabla_X Y_i, \dots, Y_k).$$

We shall not prove this, just multilinear algebra. (Highly recommended to waste several sheets of paper and hours of our life trying to get through this jungle of endless formulas, trying to do the proof.) □

Def. For $F \in \Gamma^\infty(T^k_\ell M)$, the total covariant derivative is $\nabla(F) \in \Gamma^\infty(T^{k+1}_\ell M)$.

For $f \in C^\infty(M)$, $\nabla(f) = df$.

Def. $f \in C^\infty(M)$: the covariant Hessian of f is the 2-tensor $\nabla(\nabla f) = \nabla(df)$.

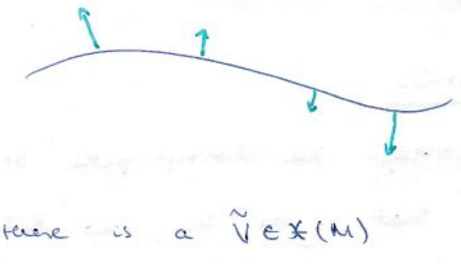
Now we return to our original goal: differentiating along a curve.

Consider $\gamma : [a,b] \to M$ smooth curve.

Def. A vector field along γ is a map $V : [a,b] \to TM$ s.t. $V(t) \in T_{\gamma(t)}M$. We denote by $T(\gamma)$ the space of all vector fields along γ.

Example. $\gamma'(t)$

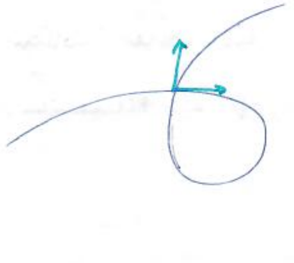
Example. $\gamma : [a,b] \to \mathbb{R}^2$, $J : \mathbb{R}^2 \to \mathbb{R}^2$ given by counterclockwise rotation over $\frac{\pi}{2}$. Then set $N(t) := J(\gamma'(t)) = (-\gamma_2'(t), \gamma_1'(t))$ is a vector field along γ normal to γ.



Example. $\tilde{X} \in \mathfrak{X}(M)$, set $X(t) := \tilde{X}_{\gamma(t)}$ is a vector field along γ.

Def. A vector field $V \in T(\gamma)$ is extendible if there is a $\tilde{V} \in \mathfrak{X}(M)$ with $V(t) = \tilde{V}_{\gamma(t)}$.

Not all elements of $T(\gamma)$ are extendible: if $\gamma(t_0) = \gamma(t_1)$ but $\gamma'(t_0) \neq \gamma'(t_1)$, then γ' is not extendible.



Covariant derivative of vector fields along a curve

Lemma. ∇ a linear connection on M. For every smooth curve $\gamma: [a,b] \to M$, ∇ determines a unique operator $D_t : T(\gamma) \to T(\gamma)$ satisfying

a) $D_t\,(aV + bW) = a\,D_t V + b\,D_t W \quad \forall a,b \in \mathbb{R}$

b) $D_t\,(f \cdot V) = f' \cdot V + f\,D_t V \quad \forall f \in C^\infty([a,b])$

c) If V is extendible, then for any extension $\tilde{V} \in \mathfrak{X}(M)$: $D_t V = \nabla_{\gamma'(t)} \tilde{V}$.

Def. $D_t V$ is called the covariant derivative of V along γ.

PROOF: Uniqueness: As before, we can show that $D_{t_0} V$ only depends on the values of V in a small interval around t_0: $(t_0 - s,\ t_0 + s)$, $s > 0$.

Then near $\gamma(t_0)$ we can write

$$V(t) = \sum_j V_j(t) \frac{\partial}{\partial x_j}$$

in coordinates.

$\frac{\partial}{\partial x_j}$ is extendible (locally every vector field is extendible, because the smoothness guarantees that some time passes between self-intersections)

$$\Rightarrow D_t V(t_0) = \sum_j V_j'(t_0) \frac{\partial}{\partial x_j} + V_j(t_0)\, \nabla_{\gamma'(t_0)} \frac{\partial}{\partial x_j}$$

$$= \sum_k \left(V_k'(t_0) + \sum_{i,j} V_j(t_0) \cdot \gamma_i'(t_0)\, \Gamma_{ij}^k(\gamma(t_0)) \right) \frac{\partial}{\partial x_k} \qquad (*)$$

This shows the uniqueness.

If $\gamma([a,b]) \subset U$ is contained in a single chart U, we define $D_t V$ by $(*)$. In general, define $D_t V$ on charts by $(*)$ and check that the definition agrees on overlaps.

Geodesics

Intuitively: the shortest path between two points.

Note that thus far, no Riemannian metric has been used in the discussion about connections and covariant derivatives. The notion of a geodesic relies on the notion of a connection, hence it is independent from the concept of a Riemannian metric.

Global Analysis I, lecture 14 – 15

∇ a linear connection on M

$\gamma : [a,b] \to M$

Def. The acceleration of γ is the vector field $D_t \gamma'$ along γ.

(This is the analogue to the second derivative.)

Def. A smooth curve γ is a geodesic if $D_t \gamma' \equiv 0$.

Theorem. (Existence and uniqueness of geodesics)

∇ a linear connection on a manifold M.

For any $p \in M$, $V \in T_pM$ and $t_0 \in \mathbb{R}$, there exist an open interval $I \subset \mathbb{R}$, $t_0 \in I$ and a geodesic $\gamma : I \to M$ such that $\gamma(t_0) = p$ and $\gamma'(t_0) = V$

Any two such geodesics agree on their common domain.

PROOF. Choose coordinates x_i around p.

A curve γ is a geodesic iff its component functions in these coordinates satisfy the so-called geodesic equation: $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$

$$\left(\gamma_k''(t) + \sum_{i,j} \gamma_i'(t)\, \gamma_j'(t)\, \Gamma_{ij}^{k}(\gamma(t))\right) = 0 \qquad \forall k = 1, \dots, n$$

This is a second order system of ODEs for $\gamma_i(t)$.

It can be converted into the first order system:

$$\gamma_k'(t) = v_k(t)$$

$$v_k'(t) = \sum_{i,j} - v_i(t)\, v_j(t)\, \Gamma_{ij}^{k}(\gamma(t))$$

"We are using a heavier hammer from a course that hopefully someone took. I mean, at some point, we have to outsource work."

By the existence and uniqueness of solutions theorem for ODEs, for any pair (p, V) there is an $\varepsilon > 0$ and a unique solution $\eta : (t_0 - \varepsilon, t_0 + \varepsilon) \to U \times \mathbb{R}^n$ satisfying $\eta(t_0) = (p, V)$.

Writing $\eta(t) = (\dots, \gamma_i(t), \dots, v_j(t), \dots)$, the components γ_i give $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$, satisfying the conditions. This proves the existence.

Uniqueness: take geodesics $\gamma, \sigma : I \to M$ on $I = (t_0 - \varepsilon, t_0 + \varepsilon)$ with $\sigma(t_0) = \gamma(t_0)$ and $\sigma'(t_0) = \gamma'(t_0)$. The uniqueness of solutions for ODEs gives that $\sigma = \gamma$ on some nbh. $(t_0 - \delta, t_0 + \delta)$.

$$\text{Continuity} \Rightarrow \left.\begin{aligned} \gamma(t_0 \pm \delta) &= \sigma(t_0 \pm \delta) \\ \gamma'(t_0 \pm \delta) &= \sigma'(t_0 \pm \delta) \end{aligned}\right\} \Rightarrow \gamma = \sigma \text{ on a slightly bigger interval.}$$

by the same argument $\Rightarrow \sigma = \gamma$ on I. □

Corollary. $\forall p \in M$, $V \in T_pM$: there is a unique maximal geodesic $\gamma : I \to M$ with $0 \in I$, $\gamma(0) = p$, $\gamma'(0) = V$.

Here maximal means that γ cannot be extended to any larger interval.

This geodesic is denoted γ_V. □

Parallel translation

5.12.2017
(S. Roos, who uses different notations than BH)

M mfld, ∇ lin. connection

Def. A vector field V is parallel along a curve $\gamma : I \to M$ if $D_t V \equiv 0$.

Rmk. $D_t V$ is often denoted as $\nabla_{\dot\gamma} V$

A geodesic γ is characterised by $D_t \dot\gamma = 0$.

Def. A vector field X on M is parallel if X is parallel along every curve, or equivalently, if $\nabla X = 0$.

Theorem. Given a curve $\gamma : I \to M$ and $V_0 \in T_{\gamma(t_0)}M$ for some $t_0 \in I$, then there is a unique parallel vector field V along γ s.t. $V(t_0) = V_0$.

Theorem. $I \subset \mathbb{R}$ an interval, and for $1 \le j, k \le n$ let $A_j^k : I \to \mathbb{R}$ be
(PO)
smooth functions.

The linear initial value problem

$$\begin{cases} \dot V^k(t) = A_j^k(t)\, V^j(t) \\ V^k(t_0) = B^k \end{cases}$$

has a unique solution on the whole I for any $t_0 \in I$ and any initial vector $(B^1, \dots, B^n)$.

PROOF OF THE ABOVE THM:

- Suppose $\gamma(I)$ is contained in a single chart domain.

$$0 = D_t V(t_0) = \dot V(t_0)\,\partial_j + V^j(t_0)\, \nabla_{\dot\gamma(t_0)} \partial_j$$

$$= \Big(\dot V^k(t_0) + V^j(t_0)\, \underset{i^{\text{th}}\text{ component}}{\dot\gamma^i(t_0)}\, \Gamma_{ij}^k(\gamma(t_0))\Big)\partial_k$$

V is hence parallel iff:

$$\dot V^k(t) = -V^j(t)\, \dot\gamma^i(t)\, \Gamma_{ij}^k(\gamma(t)),$$

and by the Thm. about the l.i.v.p. $\exists!$ a parallel V with $V(t_0) = V_0$.

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- If $\gamma(I)$ is not contained in a single chart domain:

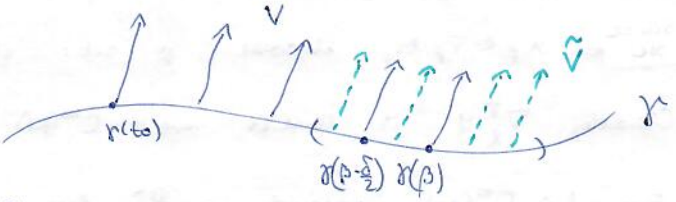
↯ Assume a parallel V does not exist on the whole I.

Set $\beta := \sup \{ b > t_0 \mid \exists!$ parallel vector field on $[t_0, b] \} > t_0$

If $\beta \in I$, choose coordinates on $\gamma((\beta-\delta, \beta+\delta))$ for some small $\delta > 0$.

By the Thm. about lin. i.v.p. there is a unique $\tilde{V}$ on $(\beta-\delta, \beta+\delta)$ s.t. $\tilde{V}(\beta - \frac{\delta}{2}) = V(\beta - \frac{\delta}{2})$.

By uniqueness, $V = \tilde{V}$ on $(\beta-\delta, \beta)$.



Hence V extends to $(\beta, \beta+\delta)$ by $\tilde{V}$, contradicting the def of β and the assumption $\beta \in I$. ↯ □

Def. γ curve, $t_0, t_1 \in I$.

$$\underline{P_{t_0 t_1}} : T_{\gamma(t_0)} M \longrightarrow T_{\gamma(t_1)} M$$
$$V_0 \longmapsto V(t_1)$$

where V is the unique par. vct. field with $V(t_0) = V_0$.

P_{t_0, t_1} is the parallel translation operator. (It takes a vct. field V_0 and moves it parallelly.)

P_{t_0, t_1} defines an isomorphism. $T_{\gamma(t_0)} M \to T_{\gamma(t_1)} M$.

$$D_t V(t_0) = \lim_{t \to t_0} \frac{P^{-1}_{t_0, t_1} V(t) - V(t_0)}{t - t_0}$$

Riemannian connection (Levi-Civita connection)

A natural connection on Riem. manifolds.

(M, g) a Riem. manifold

Theorem. (Nash embedding theorem) Any m-dimensional Riemannian manifold (M, g) can be isometrically embedded in $\mathbb{R}^{N(m)}$.

Let $(M,g) \hookrightarrow \mathbb{R}^n$, $\pi^T: T\mathbb{R}^n \to TM$ projection, $\bar{\nabla}$ the eucl. conn. in $\mathbb{R}^n$.

Define $\nabla^T: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $\nabla^T_X Y := \pi^T(\bar{\nabla}_X Y)$

Lemma. ∇^T is a well-def. conn. on M.

Pf: By construction, $\nabla^T_X Y$ is independent from the choice of extending X.

$\bar{\nabla}_X Y$ at p depends only on values of Y along a curve γ with $\gamma(0) = p$, $\dot{\gamma}(0) = X$.

Since $X_p \in T_pM$, choose γ s.t. it maps into M.

Clearly $\nabla^T_X Y$ is linear over $C^\infty(M)$ in X and over $\mathbb{R}$ in Y.

For $f \in C^\infty(M)$, extend to $\mathbb{R}^n$ by

$$\nabla^T_X (fY) = \pi^T\left(\bar{\nabla}_X (fY)\right) = X(f)\, \pi^T Y + f\, \pi^T(\bar{\nabla}_X Y) =$$
$$= X(f)\, Y + f\, \nabla^T_X Y. \qquad \square$$

Def. (M,g) a Riemannian manifold, ∇ a lin. connection.

∇ is compatible with g if $\forall X, Y, Z \in \mathfrak{X}(M)$:

$$\nabla_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

∇ is also called a metric connection.

Example. On $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, $\bar{\nabla}$ is metric.

For $(M,g) \hookrightarrow \mathbb{R}^n$, ∇^T is metric.

Lemma. ∇ a connection on (M,g). TFAE:

a) ∇ is metric

b) $\nabla g \equiv 0$

c) If V, W are vct. fields along a curve γ, then

$$\frac{d}{dt}\langle V, W\rangle = \langle D_t V, W\rangle + \langle V, D_t W\rangle.$$

d) If V, W are parallel along γ, then $\langle V, W\rangle$ is constant.

e) The parallel translation $P_{t_0 t_1}: T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$ are isometries.

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PROOF: a) ⇔ b): $\nabla_X \left(g(Y,Z) \right) = (\nabla_X g)(Y, \overset{Z}{\cancel{X}}) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

g is a metric, hence a tensor, and $\nabla_X g$ is its covariant derivative

b) ⇔ c): same, just along curves.

d) ⇔ e): trivial

c) ⇒ d): $\frac{d}{dt} \langle V, W \rangle = \langle D_t V, W \rangle + \langle V, D_t W \rangle = 0$

d) ⇒ b): $\underbrace{\frac{d}{dt}\left(g(V,W) \right)}_{0} = \underbrace{\left(\frac{d}{dt} g \right)(V,W)}_{0} + \underbrace{g(D_t V, W) + g(V, D_t W)}_{\Rightarrow \; 0}$ □

If ∇^0 is metric, $A: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ skew-symmetric ⇒ $\nabla^0 + A$ is metric as well, so there is a whole family of metric connections belonging to ∇^0.

Def. Torsion tensor: $\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$.

Def. ∇ is symmetric (torsion-free) if $\tau \equiv 0$, i.e. $\tau(X,Y) = 0 \quad \forall X, Y$.

Examples. $\mathbb{R}^n$ with $\bar{\nabla}$, ∇^T.

Theorem. (Fundamental lemma of Riemannian geometry)

(M,g) a Riemannian manifold, then there exists a metric torsion-free unique linear connection ∇ on M.

This ∇ is called the Levi-Civita.

PROOF. Uniqueness. Suppose ∇ is such a connection, $X, Y, Z \in \mathfrak{X}(M)$.

$X(\langle Y, Z \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$

$Y(\langle Z, X \rangle) = \langle \nabla_Y Z, X \rangle + \langle Z, \nabla_Y X \rangle$

$Z(\langle X, Y \rangle) = \langle \nabla_Z X, Y \rangle + \langle Y, \nabla_Z X \rangle$

∇ is symmetric ⇒ $X(\langle Y, Z \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_Z X \rangle + \langle Y, [X, Z] \rangle$ (I)

$Y(\langle Z, X \rangle) = \langle \nabla_Y Z, X \rangle + \langle Z, \nabla_X Y \rangle + \langle X, [Z, Y] \rangle$ (II)

$Z(\langle X, Y \rangle) = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Y Z \rangle + \langle X, [Z, Y] \rangle$ (III)

(I) + (II) − (III): Koszul formula

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \Big(X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle - \langle Y, [X, Z] \rangle - \langle Z, [Y, X] \rangle + \langle X, [Z, Y] \rangle \Big)$$

The rhs contains no connections ⇒ ∇ is unique.

Existence $(U, (x_i))$ a chart, (x_i) local coordinates

Koszul formula: $\langle \nabla_{\partial_i} \partial_j, \partial_k \rangle = \frac{1}{2} \left(\partial_i (\langle \partial_j, \partial_k \rangle) + \partial_j (\langle \partial_k, \partial_i \rangle) - \partial_k (\langle \partial_i, \partial_j \rangle) \right)$

Set $g_{ij} := \langle \partial_i, \partial_j \rangle$, $\nabla_{\partial_i} \partial_j := \Gamma_{ij}^k \partial_k$. Let g^{kl} be the entries of the inverse matrix of $(g_{ij})_{i,j}$.

$\Rightarrow \Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} + \partial_k g_{ij})$

These Christoffel symbals define a connection.

Need to check: metric, torsion-free, linear.

$$\left.\begin{aligned} &0 = [\partial_i, \partial_j] \\ &\nabla_{\partial_i} \partial_j - \nabla_{\partial_j} \partial_i = \left(\Gamma_{ij}^k - \Gamma_{ji}^k \right) \partial_k = 0 \end{aligned}\right\} \Rightarrow \nabla \text{ is symmetric.}$$

(It suffices to check only for the ∂_i.)

$$(\partial_k g)(\partial_i, \partial_j) = \partial_k (g_{ij}) - \sum_{l=1}^{n} \left(\Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{ij} \right)$$

$$= \partial_k (g_{ij}) - \frac{1}{2} \sum_{l,m} \Big(\overbrace{g^{ml} g_{lj}}^{\delta_{mj}} (\partial_k g_{im} + \partial_i g_{km} - \partial_m g_{ki})$$

$$+ \underbrace{g^{ml} g_{li}}_{\delta_{mi}} (\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{ij}) \Big)$$

$\square$

Convention. Unless stated otherwise, ∇ denotes the Levi-Civita connection.

Rmk. $(M, g) \hookrightarrow \mathbb{R}^n$ isometrically embedded, then ∇^T is the Levi-Civita connection.

Lemma. All geodesics have constant speed.

Pf: γ geodesic $\Leftrightarrow D_t \dot{\gamma} = 0 \Leftrightarrow \frac{d}{dt} \langle \dot{\gamma}, \dot{\gamma} \rangle = 2 \langle D_t \dot{\gamma}, \dot{\gamma} \rangle = 0$. $\square$

Prop. (PO) Let $\varphi : (M, g) \longrightarrow (\tilde{M}, \tilde{g})$ be an isometry, ∇ and $\tilde{\nabla}$ the respective Levi-Civita connections.

a) $\varphi_* (\nabla_X Y) = \tilde{\nabla}_{\varphi_* X} (\varphi_* Y)$

b) $\gamma : I \to M$ a curve, $\tilde{\gamma} := \varphi \circ \gamma$. Then $\varphi_* D_t V = \tilde{D}_t (\varphi_* V)$

c) If γ is geodesic in M with $\dot{\gamma}(0) = V$, $\gamma(0) = p$, then $\tilde{\gamma} = \varphi \circ \gamma$ is a geodesic in $\tilde{M}$ with $\tilde{\gamma}(0) = p$, $\dot{\tilde{\gamma}}(0) = \varphi_* V$.

This will be an exercise. Hint: define a pullback connection

$$(\varphi^* \tilde{\nabla})_X Y = \varphi_*^{-1} \left(\tilde{\nabla}_{\varphi_* X} (\varphi_* Y) \right).$$

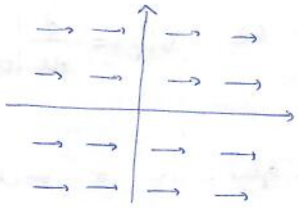
Check that this is metric and torsion-free, hence it coincides with the Levi-Civita connection ∇.

Integral curves

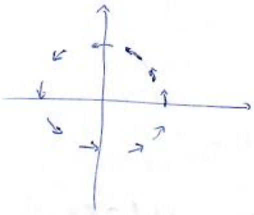
Def. V vector field on M, $I \subseteq \mathbb{R}$ interval, $\gamma : I \to M$ smooth curve is an integral curve if $\dot{\gamma}(t) = V(\gamma(t))$.

If $0 \in I$ then $\gamma(0)$ is the starting point of γ.

Example. $V = \partial_x$ on $\mathbb{R}$ $\longrightarrow$ the integral curves are $\gamma(t) = (a+t, b)$ for some $a, b \in \mathbb{R}$



$W = x\partial_y - y\partial_x =$ "ix" $\longrightarrow$ $\gamma(t) = (a\cos t - b\sin t,\ a\sin t + b\cos t)$



Lemma. (Translation lemma.) V smooth vector field, $I \subseteq \mathbb{R}$ open interval, $a \in \mathbb{R}$, γ int. curve of V $\Rightarrow$ $\tilde{\gamma} : I + a \to M$, $t \mapsto \gamma(t-a)$ is integral.

PF: easy. □

Let V be a vect. field.
Assume that $\forall p \in M$ has a unique integral curve $\Theta^p : \mathbb{R} \to M$ that starts at p.

Define $\Theta^t : M \to M$

$$p \mapsto \Theta^p(t)$$

If $q := \Theta^p(s)$, then the trans. lemma implies that $t \mapsto \Theta^p(t+s)$ is an integral curve starting at q.

$$\Theta_t \circ \Theta_s(p) = \Theta_{t+s}(p), \qquad \Theta_0(p) = p.$$

$\Rightarrow$ Θ gives a group action of $(\mathbb{R}, +)$

Def. A global flow / one-parameter grp. action is the left $\mathbb{R}$-action on M.

We call the action smooth if $\Theta: \mathbb{R} \times M \to M$ is smooth.

$\Theta_t(p) = t.p$

Θ_t is a homeomorphism / diffeomorphism.

Def. Parametrised orbit of p: $\Theta^p: \mathbb{R} \to M$, $t \mapsto \Theta_t(p)$

We define the infinitesimal generator of Θ to be $V(p) = \frac{d}{dt}\Big|_{t=0} \Theta^p(t)$.
(if Θ is smooth)

Prop. $\Theta: \mathbb{R} \times M \to M$ smooth global flow. The infin. gen. is a vector field and Θ^p is an integral curve of the generator.

Pf: ∂_t vct field on $\mathbb{R} \times M$

$$V = \Theta_*(\partial_t)\Big|_{\{0\}\times M}$$

$$\Theta_*(\partial_t)(t_0, p) = \frac{d}{dt}\Big|_{t=0} \Theta(t_0 + t, p) = \frac{d}{dt}\Big|_{t=0} \Theta(t, \Theta_{t_0}(p)) = V(\Theta^p(t_0)). \qquad \square$$

Example.
$V = \partial_x$ is the generator of the flow $\Theta_t(x,y) = (x+t, y)$

$W = x\partial_y - y\partial_x$ is the gen. of $\Theta_t(x,y) = (x\cos t - y\sin t, \; x\sin t - y\cos t)$.

Fundamental thm of flows

$V = \partial_x$ defined on $\mathbb{R}^2 \setminus \{0\}$. $\rightarrow$ there is no global flow def. by V.

Def. $D \subseteq \mathbb{R} \times M$ open subset s.t. $\underline{D^p} = \{(t,q) \in D \mid q = p\}$ is an open interval containing 0.

A smooth flow on M is a smooth map $\Theta: D \to M$ s.t.
$\Theta(0,p) = p$, $s \in D^p$, $t \in D^{\Theta(s,p)}$ and $t+s \in D^p$ then $\Theta(t, \Theta(s,p)) = \Theta(t+s, p)$

We can again define an inf. generator.

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Prop. Θ is a flow, then the inf. gen. is a smooth manifold and Θ^p is an int. curve.

Pf: As above. (We have not used globality there.) □

Theorem (Fundamental thm. of flows)

V a smooth vector field on M.

Then there exists a unique maximal smooth $\Theta : D \to M$ whose inf. generator is V and

a) $\forall p$: Θ^p is the unique maximal integral curve of V starting at p.

b) if $s \in D^p$ then $D^{\Theta(s,p)} = D^p - s$.

c) $\forall t \in \mathbb{R}$: $M_t = \{ p \in M \mid (t,p) \in D \}$ is open in M and $\Theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse Θ_{-t}.

d) $\forall (t,p) \in D$: $(\Theta_t)_* (V(p)) = V(\Theta_t(p))$.

Theorem. (ODE Existence, uniqueness and smoothness)

$U \subset \mathbb{R}^n$ open, $V : U \to \mathbb{R}^n$ a smooth map.

For $t_0 \in \mathbb{R}$ and $x \in U$ consider the following i.v.p.:

$$(\dot{\gamma}^i)(t) = V^i(\gamma(t)), \quad \gamma^i(t_0) = x^i.$$

a) Existence: $\forall t_0 \in \mathbb{R}$ $\forall x_0 \in U$ $\exists$ a unique interval $I_0 \subseteq \mathbb{R}$ s.t. $t_0 \in I_0$ and $\exists$ an open set $U_0 \subseteq U$ s.t. $x_0 \in U_0$ and $\forall x \in U_0$ there is a smooth curve starting at x and solving the ivp.

b) Uniqueness: any two diffable solutions of the ivp agree on their common domain.

c) Smoothness: $\Theta : I_0 \times U_0 \to U$, $\Theta(t,x) = \gamma_x(t)$ where γ_x is the sol. of the i.v.p. starting at x. Then Θ is smooth.

PF OF THE FUND. THM:

Let $\gamma, \tilde{\gamma}$ be two i. curves of V, starting at x_0.

ODE b) $\Rightarrow$ the set on which $\gamma = \tilde{\gamma}$ is open.

But this set is also closed $\Rightarrow$ $\gamma = \tilde{\gamma}$ on their common domain.

$D^p :=$ union of all intervals on which the int. curve starting at p is defined

$$D := \bigcup_{p \in M} D^p \times \{p\}$$

$\Theta(t, p) := \Theta^p(t)$

Fix any $\underline{p \in M}$ and $\underline{s \in D^p}$. Let $q := \Theta(s, p)$

$$\gamma : D^p \setminus \{s\} \longrightarrow M \qquad \text{is an int. curve through } q$$
$$t \longmapsto \gamma(t+s)$$

$\Rightarrow$ Θ is a flow

The domain of Θ^q cannot be larger than D^q $\Rightarrow$ $D^p - s \subseteq D^q$

$0 \in D^p \Rightarrow -s \in D^q, \ \Theta^q(-s) = p \Rightarrow D^q + s \subseteq D^p \Rightarrow D^q \subseteq D^p - s$

$\Rightarrow$ part b) ✓

ODE a) $\Rightarrow$ D open in $\mathbb{R} \times M$

ODE c) $\Rightarrow$ Θ is smooth.

D is open $\Rightarrow$ M_t is open & group laws of Θ $\Rightarrow$ part c) ✓

$\Theta_{t_0}(p) = q$. Wts: $(\Theta_{t_0})_*(V(p)) = V(q)$

We apply $(\Theta_{t_0})_*(V(p))\, f = V_p(f \circ \Theta_{t_0}) = \left.\frac{d}{dt}\right|_{t=0} f \circ \Theta_{t_0} \circ \Theta^p(t) =$

$$= \left.\frac{d}{dt}\right|_{t=0} f(\Theta_{t_0+t}(p)) = \left.\frac{d}{dt}\right|_{t=0} f(\Theta^q(t)) = V(q)\, f$$

$\Rightarrow$ part d) □

PROOF OF THE ODE THM:

$U \subseteq \mathbb{R}^n$ open, $V: U \longrightarrow \mathbb{R}^n$ is Lipschitz continuous,

and $\forall t_0 \in \mathbb{R}$ $\forall x \in U$ we consider the ivp

$$\frac{d}{dt}\left(\gamma^i\right)(t) = V^i(\gamma(t))$$

A smooth vector field is locally Lipschitz continuous.

Lemma. (Gronwall) Suppose $J_0 \subseteq \mathbb{R}$ open interval, $t_0 \in J_0$,

$u: J_0 \longrightarrow \mathbb{R}^n$ diffable. and

$$|u'(t)| \leq A\,|u(t)| + B$$

for $A, B \geq 0$, $\forall t \geq J_0$.

Then

$$|u(t)| \leq e^{A|t-t_0|}\,|u(t_0)| + \frac{B}{A}\left(e^{A|t-t_0|} - 1\right)$$

Pf: $J_0^t := \{t \geq t_0$

Substitute $\tilde{t} \leq t_0$ by $t_0 - \tilde{t} = t - t_0$.

If $|u(t)| > 0$ then $|u(t)|$ is diffable:

$$\frac{d}{dt}\,|u(t)| = \frac{d}{dt}\,(u(t)\cdot u(t))^{1/2} = \frac{1}{2}\,(u(t)\cdot u(t))^{-1/2}\cdot 2\,(u(t)\cdot u'(t))$$

$$\leq \frac{1}{2}\,|u(t)|^{-1}\cdot 2\,|u(t)|\cdot|u'(t)| = |u'(t)| \leq$$

$$\leq A\,|u(t)| + B$$

$g: J_0^t \longrightarrow \mathbb{R}$ def. by $g(t) := e^{A(t-t_0)}\,|u(t_0)| + \frac{B}{A}\left(e^{A(t-t_0)} - 1\right)$

$g(t_0) = |u(t_0)|$. $g(t) > 0$ $\forall t > t_0$ and $g'(t) = A\,g(t) + B$

$f: J_0^t \longrightarrow \mathbb{R}$ by $f(t) := e^{-A(t-t_0)}\left(|u(t)| - g(t)\right)$

$f(t) = 0$ and the lemma is equivalent to $f(t) \leq 0$.

If $f(t) > 0 \Rightarrow |u(t)| > 0 \Rightarrow f$ diffable.

$$f'(t) = -A\,e^{-A(t-t_0)}\left(|u(t)| - g(t)\right) + e^{-A(t-t_0)}\left(\frac{d}{dt}|u(t)| - g'(t)\right) \leq$$

$$\leq -A\,e^{-A(t-t_0)}\left(|u(t)| - g(t)\right) + e^{-A(t+t_0)}\left(A\,|u(t)| + B - A\,g(t) - B\right) \leq 0.$$

Suppose $\exists t_1 \in \mathcal{J}_0^t$ s.t. $f(t_1)>0$

Define $\tau = \sup \{ t \in [t_0, t_1] \mid f(t) < 0 \}$.

$\Rightarrow f(\tau) = 0, \quad f(t) > 0 \quad \forall t \in (\tau, t_1]$

Mean value thm $\Rightarrow f'(t) > 0$ for some $(\tau, t_1]$. ↯ $\boxed{L}$

Thm. (Existence and uniqueness of ODE solutions)

$U \subseteq \mathbb{R}^n$ open, $V: U \to \mathbb{R}^n$ Lipschitz, $(t_0, x_0) \in \mathbb{R} \times U$.

Then there is a $\mathcal{J}_0 \subseteq \mathbb{R}$ interval, $t_0 \in \mathcal{J}_0$, $U_0 \subseteq U$ open, $x \in U_0$

and for each $x \in U_0$ a C^1-curve $\gamma: \mathcal{J}_0 \to U$ satisfying the ivp.

If we have two solutions, then they agree on their common domain.

Thm. (Banach fixed point thm.)

(X, d) complete metric space, $T: X \to X$ contraction (i.e. T is Lipschitz with Lipschitz constant < 1). Then T has a unique fixed point.

Pf: Exercise. $\square$

PF OF E&U THM: Suppose γ solves the ivp. Fund. thm. of calculus $\Rightarrow$ γ is C^1

$$\Rightarrow \gamma^i = x^i + \int_{t_0}^{t} V^i(\gamma(s))\, ds$$

Conversely, if γ has this property, then γ is a C^1 solution of the ivp.

Suppose $\mathcal{J}_0$ is an open interval containing t_0.

For any cont. curve γ starting at x define

$$I\gamma: \mathcal{J}_0 \to \mathbb{R}^n, \qquad I\gamma(t) := x + \int_{t_0}^{t} V(\gamma(s))\, ds$$

V is Lipschitz $\Rightarrow \exists C$ s.t. $|V(x) - V(\tilde{x})| \leq C \cdot |x - \tilde{x}|$.

Let M denote the supremum of $|V|$ on $\overline{B}_r(x_0) \subseteq U$ for some $r > 0$.

Choose $\delta > 0$, $\varepsilon > 0$ small s.t. $\delta < \frac{r}{2}$ and $\varepsilon < \min\left(\frac{r}{2M}, \frac{1}{2}\right)$.

$\mathcal{J}_0 = (t_0 - \varepsilon, t_0 + \varepsilon)$ and $U_0 = B_\delta(x_0)$

Let $\mathcal{M}_x$ denote the space of cont ~~functions~~ curves $\gamma: \mathcal{J}_0 \to \overline{B}_r(x_0)$ s.t.

$\gamma(t_0) = x, \quad d(\gamma, \tilde{\gamma}) = \sup_{t \in \mathcal{J}_0} |\gamma(t) - \tilde{\gamma}(t)|$.

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In $\mathcal{M}_x$, every Cauchy sequence is uniformly Cauchy.

Hence every Cauchy sequence converges to a cont. map, i.e. $\mathcal{M}_x$ is complete.

Want to define: $I : \mathcal{M}_x \to \mathcal{M}_x$ as defined before.

Well-definedness: $I\gamma$ is $\overset{\text{continuous}}{\text{~~well-def~~}}$, $I\gamma(t_0) = x$, $I\gamma(\mathcal{J}_0) \subseteq \overline{B}_r(x_0)$

$$\left| I\gamma(t) - x_0 \right| = \left| x + \int_{t_0}^{t} V(\gamma(s))\, ds - x_0 \right| \leq |x - x_0| + \int_{t_0}^{t} V(\gamma(s))\, ds$$

$$< \delta + M\varepsilon < r$$

If $\gamma, \tilde{\gamma} \in \mathcal{M}_x$ then

$$d(I\gamma, I\tilde{\gamma}) = \sup \left| \int_{t_0}^{t} V(\gamma(s))\, ds - \int_{t_0}^{t} V(\tilde{\gamma}(s))\, ds \right| \leq$$

$$\leq \sup \int_{t_0}^{t} \left| V(\gamma(s)) - V(\tilde{\gamma}(s)) \right| ds \leq$$

$$\leq \sup C \cdot \int_{t_0}^{t} |\gamma(s) - \tilde{\gamma}(s)|\, ds \leq C \cdot \varepsilon \cdot d(\gamma, \tilde{\gamma})$$

By our choice of ε, I is a contraction.

By the Banach F.p. thm., the ODE has a unique solution. ⊠

Suppose $\gamma, \tilde{\gamma}$ both solve the i.v.p.

$$\Rightarrow \left| \frac{d}{dt}\Big|_{t=0} \left(\tilde{\gamma}(t) - \gamma(t) \right) \right| = \left| V(\tilde{\gamma}(t)) - V(\gamma(t)) \right| \leq C \cdot |\tilde{\gamma}(t) - \gamma(t)|.$$

Gronwall ineq. $\Rightarrow$ $|\tilde{\gamma}(t) - \gamma(t)| \leq e^{C \cdot |t - t_0|} \, |\tilde{\gamma}(t_0) - \gamma(t_0)|$.

$\gamma(t_0) = \tilde{\gamma}(t_0) \Rightarrow \gamma = \tilde{\gamma}$.

□

13.12.2017

Now we prove c). To recap:

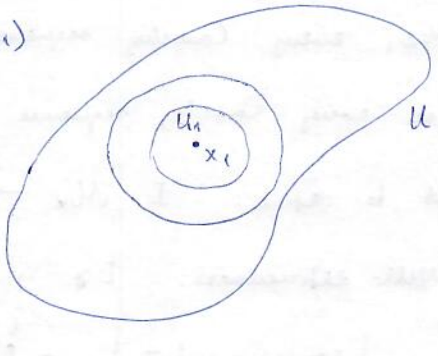
> t_0, x_0 as before, define $\Theta : \mathcal{J}_0 \times U_0 \longrightarrow U$ where $\gamma_x(t)$ is a solution
> $(t, x) \longmapsto \gamma_x(t)$
> to the i.v.p. with t_0. $\Rightarrow$ Θ is smooth.

Hard part: continuity and diffability. Easier: $C^k \to C^{k+1}$ induction.

Continuity of Θ: Take $\mathcal{J}_0 \times U_0 \ni (t_1, x_1)$ and show that Θ is cont. at (t_1, x_1).

Let $\mathcal{J}_1 \subset \mathcal{J}_0$ be an bounded open interval, with $\overline{\mathcal{J}_1} \subset \mathcal{J}_0$, $t_0, t_1 \subset \mathcal{J}_1$

Choose $r > 0$ s.t. $\overline{B_{2r}(x_1)} \subset U_0$, and $U_1 := B_r(x_1)$



Let C be a Lipschitz constant for V and define

$$M := \sup_{\overline{U_1}} |V| \qquad T := \sup_{\overline{\mathcal{J}_1}} |t - t_0|$$

We will show the continuity of Θ on $\overline{\mathcal{J}_1} \times \overline{U_1}$.

From the uniqueness of solutions proof we get

$$\left|\Theta(t, x) - \Theta(t, \tilde{x})\right| \leq e^{CT} |\tilde{x} - x|.$$

Thus for each t: Θ is Lipschitz in x.

We need continuity in (t, x).

Take (t, x) and $(\tilde{t}, \tilde{x})$ arbitrarily. We have

$$\Theta^i(t, x) = x^i + \int_{t_0}^{t} V^i(\Theta(s, x))\, ds,$$

and we estimate

$$|\Theta(\tilde{t}, \tilde{x}) - \Theta(t, x)| \leq |x - \tilde{x}| + \left| \int_{t_0}^{\tilde{t}} V(\Theta(s, \tilde{x}))\, ds - \int_{t_0}^{t} V(\Theta(s, x))\, ds \right|$$

WLOG: $\tilde{t} \geq t$

$$\leq |x - \tilde{x}| + \int_{t_0}^{t} |V(\Theta(s, \tilde{x})) - V(\Theta(s, x))| + \int_{t_0}^{\tilde{t}} |V(\Theta(s, \tilde{x}))| ds$$

$$\leq |\tilde{x} - x| + c \int_{t_0}^{t} |\Theta(s, \tilde{x}) - \Theta(s, x)| ds + \int_{t}^{\tilde{t}} M\, ds$$

$$\leq |\tilde{x} - x| + CT\, e^{CT} |\tilde{x} - x| + M |\tilde{t} - t|,$$

which implies continuity. ✓

C^1-ness of Θ: Def. $\mathcal{J}_1$ and U_1 as before. Write the c.o.p. as

$$\frac{d}{dt} \Theta^i(t, x) = V^i(\Theta(t, x)) \qquad \Theta^i(t, x) = x^i$$

Since Θ is continuous, this implies that the t-derivative of Θ exists and is continuous in ~~both~~ (t, x).

Define the differential quotient:

$$(\Delta_h)^i_j : \overline{\mathcal{J}_1} \times \overline{U_1} \to \mathbb{R}$$

$$(t, x) \mapsto \frac{\Theta^i(t, x + h e_j) - \Theta^i(t, x)}{h}$$

Global Analysis, lecture 17

We can view Δ_h as a function $\Delta_h : \overline{J}_1 \overset{\times}{\cancel{m}} \overline{U}_1 \to \mathbb{R}^{n\times n}$

$$(t,x) \mapsto \left((\Delta_h)^i_j(t,x)\right)_{i,j}$$

If the limit $\lim_{h\to 0} \Delta_h(t,x)$ exists, then it must equal $\frac{\partial}{\partial x_j}\left(\Theta^i(t,x)\right)$

We will show that Δ_h converges uniformly as $h \to 0$, whence the limit exists and is continuous.

We utilise the Taylor expansion for V:

$\forall t \in \overline{J}_1,\ y \in \overline{U}_1,\ v \in B_r(0)$:

$$V^i(y+v) = \sum_k v^k \frac{\partial V^i}{\partial y_k}(y) + \underbrace{v^k \int_0^1 \left(\frac{\partial V^i}{\partial y_k}(y+sv) - \frac{\partial V^i}{\partial y_k}(y)\right) ds}_{=: G^i_k(y,v)}$$

$$= \sum_k v^k \frac{\partial V^i}{\partial y_k}(y) + G^i_k(y,v)$$

Note that if $v = 0$ then $G^i_k(y,v| = 0$.

$G^i_k(y,v)$ is defined on a compact set $\Rightarrow$ is uniformly continuous:

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|G^{\cancel{i}}_{\cancel{k}}(y,v)| < \varepsilon \quad \forall y \in \overline{U}_1 \quad \forall |v| < \delta$

where G is the matrix valued function $(G^i_k)_{i,k}$ and we use some norm $|\cdot|$ on $\mathbb{R}^{n\times n}$.

Δ_h satisfies $(\Delta_h)^i_j(t_0,x) = \delta^i_j$ because $\Theta(t_0,x) = x$.

We now compute the t-derivative of Δ_h. Define $\underset{v}{\cancel{v}} = (\underset{v}{\cancel{v}}^1, \dots, \underset{v}{\cancel{v}}^n)$ by

$$v\,\cancel{v}^k := \Theta^{\overset{k}{\cancel{k}}}(t, x+he_j) - \Theta^{\cancel{k}k}(t,x) = h\,(\Delta_h)^k_j(t,x)$$

$$\frac{d}{dt}(\Delta_h)(t,x) = \frac{1}{h}\left(\frac{d}{dt}\Theta^i(t, x+he_j) - \frac{d}{dt}\Theta^i(t,x)\right) = \frac{1}{h}\left(V^i(\Theta(t\cancel{t}, x+he_j)) - V^i(\Theta(t,x))\right) =$$

$$= \frac{1}{h}\left(\sum_k \underset{v}{\cancel{v}}^k \frac{\partial V^i}{\partial y_k}(\underbrace{\Theta(t,x)}_{=:y}) + v^k G^i_k(y,v)\right)$$

"and we can write this nightmare as follows"

$$= \sum_k \left(\frac{\partial V^i}{\partial y_k}(\underbrace{\Theta(t,x)}_{y}) + G^i_k(y,v)\right)(\Delta_h)^k_j(t,x)$$

Thus for any nonzero $h, \tilde{h} \in B_r(0)$:

$$\frac{d}{dt}\left((\Delta_h)^i_j\ (t,x)\right) = \sum_k \frac{\partial V^i}{\partial y_k}\,\theta(t,x)\ (\Delta_h)^k_j\,(t,x) - (\Delta_{\tilde{h}})^k_j\ (t,x) +$$

$\forall h,\tilde{h} \in B_r(0)\setminus\{0\}$
Recall that $y = \theta(t,x)$

$$+\ G^i_k(y,\sigma)\cdot(\Delta_h)^k_j\,(t,x) - G^i_k(y,\tilde{\sigma})\cdot(\Delta_{\tilde{h}})^k_j\ (t,x)$$

where $\tilde{\sigma}$ is defined the same way as σ, but with $\tilde{h}$ in place of h.

Now choose. $\delta \leq r$ s.t. $|G| < \varepsilon$

and let $\quad E := \sup\limits_{x\in\overline{U}_1} |DV|$

And finally, choose h and $\tilde{h}$ in a way that $|h|, |\tilde{h}| < \dfrac{\delta e^{-CT}}{n}$.

Since $\left|(\Delta_h)^i_j\right| < e^{CT}$, we have ~~for~~ $|\Delta_h| \leq n e^{CT}$

for fixed i,j

Then both σ and $\tilde{\sigma}$ are bounded by δ.

We estimate

$$\left|\frac{d}{dt}\left(\Delta_h(t,x) - \Delta_{\tilde{h}}(t,x)\right)\right| \leq E\cdot\left|\Delta_h(t,x) - \Delta_{\tilde{h}}(t,x)\right| + 2\varepsilon\cdot n\cdot e^{CT}$$

Since $\Delta_h(t_0,x) - \Delta_{\tilde{h}}(t_0,x) = 0$, the Gronwall lemma gives

$$|\Delta_h(t,x) - \Delta_{\tilde{h}}(t,x)| \leq \frac{2\varepsilon n e^{CT}}{E}\left(e^{E|t-t_0|} - 1\right) \leq$$

$$\leq 2\varepsilon n e^{CT}\cdot\frac{1}{E}\cdot\left(e^{ET} - 1\right)$$

$\varepsilon > 0$ was arbitrary $\Rightarrow$ Δ_h is uniformly Cauchy as $h \to 0$.

Hence $(\Delta_h)^i_j$ converges uniformly to a cont. function $\left(\frac{\partial}{\partial x_j}\theta^i\right)$, as desired.

$\Rightarrow \theta$ is C^1. ✓

Now suppose we have proven that C^k-ness of θ (inductive step), $k \geq 1$. Since V is smooth, $\frac{d}{dt}\theta^i$ is also C^k. Then differentiate the equation $\theta^i(t,x) = x^i + \int\limits_{t_0}^{t} V^i(\theta(s,x))\,ds$.

$$\Rightarrow \frac{\partial\theta^i}{\partial x_j}(t,x) = \delta^i_j + \int_{t_0}^{t}\sum_k \frac{\partial V^i}{\partial y_k}(\theta(s,x))\cdot\frac{\partial\theta^k}{\partial x_j}(s,x)\,ds.$$

Fund. thm. of calculus ⇒

$$\Rightarrow \frac{d}{dt}\frac{\partial \Theta^i}{\partial x_j}(t,x) = \sum_k \frac{\partial V^i}{\partial y_k}(\Theta(t,x)) \cdot \frac{\partial \Theta^k}{\partial x_j}(t,x)$$

is a diff. equation for $\frac{\partial \Theta^i}{\partial x_j}$.

Therefore the functions $\alpha_i(t) := \Theta^i(t,x)$ and $\beta^i_j(t) := \frac{\partial \Theta^i}{\partial x_j}(t,x)$ satisfy the system of diff. equations

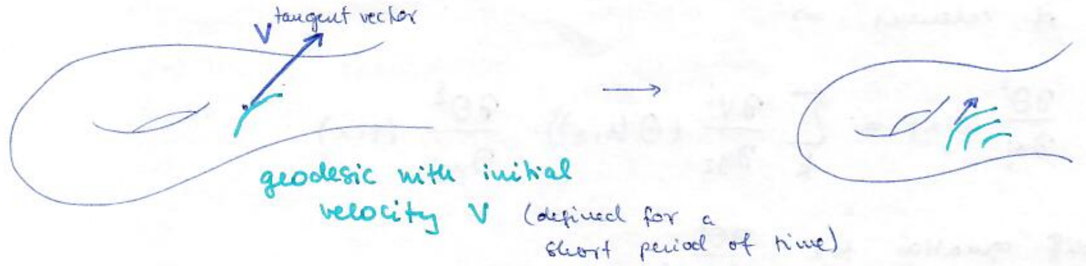
$$(**)\begin{cases} (\alpha^i)'(t) = V^i(\alpha(t)) & \alpha^i|_{t_0} = x_0 \\ (\beta^i_j)'(t) = \sum_k \frac{\partial V^i}{\partial y_k}(\alpha(t))\, \beta^k_j(t), & \beta^i_j(t_0) = \delta^i_j \end{cases}$$

Hence $(**)$ has C^k-solutions since $(**)$ is of the form of our original i.v.p., about which we have proven to have C^k-solutions

$\Rightarrow \Theta$ is C^{k+1}. ("So this inductive step is kind of a cheap trick.") □

The exponential map on a Riemannian manifold

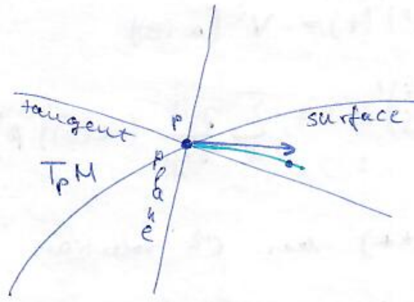
19. 12. 2017



Question: how do geodesics change if V and p vary?

exp: $TM \longrightarrow M$ not everywhere defined

Idea: for $(p, V) \in TM$ we have γ_V through p, and we wish to map (p, V) to $\gamma_V(1)$.



"Bend the plane to the surface"

exp will furnish a local identification b/w T_pM and M

To make this rigorous, set

$$\mathcal{E} := \{ V \in TM \mid \gamma_V \text{ is defined on an interval containing } [0,1] \} \subset TM$$

The map exp: $\mathcal{E} \longrightarrow M$, $V \mapsto \gamma_V(1)$ is well-defined, and is called the exponential map of (M, g).

(In the def., we use the metric and the Riem. connection, so this definition is for Riem. manifolds only.)

For $p \in M$, the restricted exponential map is $\exp_p : \mathcal{E} \cap T_pM \to M$.

We write $\mathcal{E}_p := \mathcal{E} \cap T_pM$.

Prop. (Properties of the exponential map)

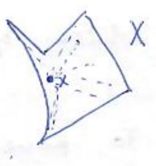
a) $\mathcal{E} \subset TM$ is an open set containing the "zero section", and each $\mathcal{E}_p$ is star-shaped around $0 \in \mathcal{E}_p$.

b) $\forall V \in TM$: $\gamma_V(t) = \exp(tV)$ whenever either side is defined

c) exp is smooth.

Global Analysis I, lecture 18

Det. A subset X of a vector space is star-shaped wrt $x \in X$ if $\forall y \in X$ the line segment connecting x and y runs entirely in X.



b) $\forall V \in TM$: $\gamma_V(t) = \exp(tV)$ whenever either side is defined.

c) exp is smooth.

Lemma. $\forall V \in TM$ $\forall c, t \in \mathbb{R}$: $\gamma_{cV}(t) = \gamma_V(ct)$ $(*)$
whenever either side is defined.

Pf: STS $\gamma_{cV}(t)$ exists and $(*)$ holds whenever $\gamma_V(ct)$ exists.
In that case $\gamma_{c \cdot \frac{1}{c} V}(ct)$ exists whenever $\gamma_V(ct)$ is defined.

Suppose that γ_V is defined on the open interval $I \subset \mathbb{R}$, $0 \in I$.
Set $\gamma_V^c(t) := \gamma_V(ct)$ which is defined on $c^{-1}I = \{t \mid ct \in I\}$.
We will show that γ_V^c is a geodesic with initial point $p = \gamma_V^c(0)$ and initial velocity cV. Then it must coincide with γ_{cV}.
By definition: $\gamma_V^c(0) = \gamma_V(0) = p$.
$\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$ in local coordinates.
Then $(\gamma_V^c)'_i(t) = \frac{d}{dt}(\gamma_V)_i(ct) = c \cdot (\gamma_V)'_i(t)$
$\Rightarrow (\gamma_V^c)'(0) = c \cdot \gamma_V'(0) = c \cdot V$.
So the only thing remaining to be shown is that γ_V is a geodesic itself.
We compute the covariant derivative D_t^c of $(\gamma_V^c)'$ along γ_V^c.
We write D_t for the covariant derivative along γ_V.

$$D_t^c (\gamma_V^c)'(t) = \sum_k \frac{d}{dt}(\gamma_V^c)_k \frac{\partial}{\partial x_k} + \sum_{i,j} \Gamma_{i,j}^k (\gamma_V^c(t)) \cdot (\gamma_V^c)'_i (\gamma_V^c)'_j \frac{\partial}{\partial x_k} =$$

$$= \sum_k c^2 (\gamma_V)''_k(ct) + \sum_{i,j} c^2 \Gamma_{i,j}^k (\gamma(ct))(\gamma_V)'_i(ct)(\gamma_V)'_j(ct) \frac{\partial}{\partial x_k} =$$

$$= c^2 D_t (\gamma_V)'(ct) = 0$$ since γ_V is a geodesic. Hence the result. □

PROOF OF PROP:

Applying the rescaling lemma for $t=1$, we get

$$\exp(cV) = \gamma_{cV}(1) = \gamma_V(c)$$

which gives b). (up to change of variables).

To prove the star-shapedness of $\mathcal{E}_p$, take $V \in \mathcal{E}_p$. So V is defined on $[0,1]$ and for $0 \leq t \leq 1$ we have $\exp(tV) = \gamma_{tV}(1) = \gamma_V(t)$, showing that the line segment tV from 0 to V is in $\mathcal{E}_p$.

Now we show that $\mathcal{E}$ is open in TM. Let $U \subset M$ be a coordinate patch with coordinates $(x_1, \dots, x_n)$ and $\pi^{-1}(U) \subset TM$ the corresponding patch for TM with coordinates $(x_1, \dots, x_n, v_1, \dots, v_n)$.

Consider the vector $G \in \mathfrak{X}(TM)$ given by

$$G_{(x,v)} := \sum_k v_k \frac{\partial}{\partial x_k} - \sum_{i,j} v_i v_j \Gamma^k_{ij}(x) \frac{\partial}{\partial v_k}$$

At the moment, G is defined on coordinate patches $\pi^{-1}(U)$.
(Similar formulas played a role at the $\exists!$ of geodesics, so this isn't a completely random definition.)

G is called the geodesic vector field on $\pi^{-1}(U)$.

The integral curves of G satisfy the following system of ODEs:

$$\begin{cases} x'_k(t) = v_k(t) \\ v'_k(t) = -\sum_{i,j} v_i(t)\, v_j(t)\, \Gamma^k_{ij}(x(t)) \end{cases}$$

which under the substitution $v_k = x'_k$ is equivalent to the geodesic equations:

$$x''_k(t) + \sum_{i,j} x'_i(t)\, x'_j(t)\, \Gamma^k_{ij}(x(t)) = 0$$

In other words, the integral curves of G on $\pi^{-1}(U)$ project to geodesics on U under the bundle projection $\pi: TM \to M$.

In coordinates: $(x(t), v(t)) \longmapsto x(t)$.

Conversely, given a geodesic $X(t)$ in U it lifts to an integral curve of G via $x(t) \mapsto (x(t), x'(t))$.

Global Analysis 1, lecture 18

Now we show that G defines a global vector field on TM, i.e. the local definitions agree on overlapping charts.

We show this by showing that $f \in C^\infty(TM)$:

$$Gf = \frac{d}{dt}\Big|_{t=0} f(\gamma_V(t), \gamma_V'(t)),$$

which is globally defined. as it is coordinate independent

$$\frac{d}{dt}\Big|_{t=0} f(\gamma_V(t), \gamma'_V(t)) = \quad \text{compute in coordinates}$$

$$= \sum_k \frac{\partial f}{\partial x_k}(x(t), v(t))\, x'_k(t) + \frac{\partial f}{\partial v_k}(x(t), v(t))\, v'_k(t)$$

$$= \sum_k \frac{\partial f}{\partial x_k}(p, V)\, V_k - \frac{\partial f}{\partial v_k}(p, V) \left(\sum_{i,j} V_i V_j \Gamma^k_{ij}(p) \right)$$

$$= Gf(p, V) \qquad \text{(we have only used the chain rule).}$$

By the ODE and Flow theorems there is an open subset $\mathcal{O} \subseteq \mathbb{R} \times TM$ containing $\{0\} \times TM$ and a smooth map

$$\theta : \mathcal{O} \longrightarrow TM$$
$$(t, (p,V)) \longmapsto \theta_{(p,V)}(t)$$

where $t \mapsto \theta_{(p,V)}(t)$ is the integral curve of G at (p,V).

For $(p,V) \in \mathcal{E}$, γ_V is defined on an open set containing $[0,1]$ and therefore θ is defined on $[0,1]$ as well. Hence $(1, (p,V)) \in \mathcal{O}$ and there is an open nbhd $\mathcal{P}$ of $(1, (p,V))$ contained in $\mathcal{O}$.

Then $\pi(\mathcal{P})$ is a nbhd of p for which the flow of G exists for $t \in [0,1]$ and exp is defined. Hence $\mathcal{E}$ is open.

Smoothness of exp: $\exp(V) = \gamma_V(1) = \pi \circ \theta(1, (p,V))$. □

Naturality of the exp map

Let $\varphi : (M, g) \longrightarrow (\tilde{M}, \tilde{g})$ be an isometry.

Then: $\cdot\ \varphi_*(\nabla_X Y) = \tilde{\nabla}_{\varphi_* X}(\varphi_* V)$ $\quad \forall X, Y \in \mathfrak{X}(M)$, $\nabla, \tilde{\nabla}$ R. connections

$\cdot\ \varphi_* D_t V = \tilde{D}_t\, \varphi_*(V)$ $\quad V$ a vector field along γ

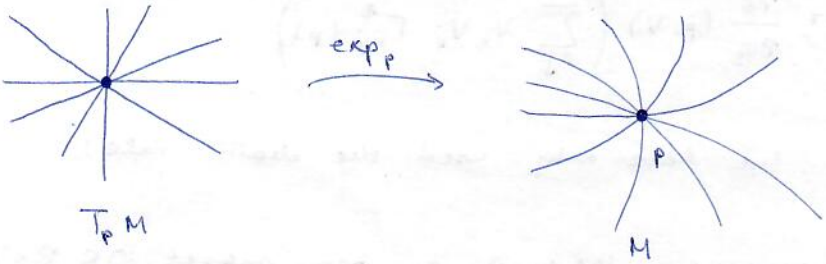
$\cdot\ \varphi$ maps geodesics to geodesics.

Prop. The diagram

$$\begin{tikzcd} T_pM \arrow[r, "\varphi_*"] \arrow[d, "\exp_p"] & T_{\varphi(p)}\tilde{M} \arrow[d, "\exp_{\varphi(p)}"] \\ M \arrow[r, "\varphi"] & \tilde{M} \end{tikzcd}$$

commutes.

Lemma: $\forall p \in M$ $\exists W$ nbhd of $0 \in T_pM$ and a nbhd. U of p s.t. $\exp_p : W \longrightarrow U$ is a diffeomorphism.



Pf: We show that the pushforward

$$(\exp_p)_* : T_0(T_pM) \longrightarrow T_pM$$

is invertible, and then the inverse function theorem proves our claim.

(Recall that T_pM is a vector space.)

Identify T_0T_pM with T_pM:

Let $V \in T_pM$ and $\tau : (-\varepsilon, +\varepsilon) \longrightarrow T_pM$ with $\varepsilon > 0$ and $\left.\frac{d}{dt}\right|_{t=0} \tau = V$, and take $\tau(t) = tV$

$$(\exp_p)_* V = \left.\frac{d}{dt}\right|_{t=0} (\exp_p \circ \tau) = \left.\frac{d}{dt}\right|_{t=0} \exp(tV) = \frac{d}{dt}\gamma_V(t) = V.$$

Hence $\exp_* : \underset{\overset{\|}{T_0T_pM}}{T_pM} \longrightarrow T_pM$ is the identity map, hence invertible.

So there exists a nbhd W of 0 for which $\exp : W \to M$ is a diffeomorphism onto its image $U := \exp(W)$. $\square$

Global Analysis I, lecture 19

20.12.2017

Def. A normal nbhd of $p \in M$ is an open set U of the form $U = \exp_p(V)$ where $V \subset T_pM$ is an open star-shaped nbhd in $0 \in T_pM$.

Recall that $\mathcal{E}_p = \mathcal{E} \cap T_pM$ is star-shaped wrt $0 \in T_pM$, so such a V always exists.

Def. A geodesic ball centered at p is a set of the form $\exp_p(B_g(0,\varepsilon))$ where $B_g(0,\varepsilon) := \{ V \in T_pM \mid \|V\|_g < \varepsilon \}$, and $\exp_p$ is defined on this ball. $B_g(0,\varepsilon)$.

Note that if ε is small enough, $\exp_p$ will be defined on $B_g(0,\varepsilon)$.

Def. A closed geodesic ball centered at p is a set of the form $\exp_p\left(\overline{B_g(0,\varepsilon)}\right)$ where there is a $V \supset \overline{B_g(0,\varepsilon)}$ open set such that $\exp_p$ is a diffeomorphism on V.

Def. A geodesic sphere is the boundary of a closed geodesic ball, i.e. a set of the form $\exp_p\left(\partial \overline{B_g(0,\varepsilon)}\right)$.

(This is because $\exp_p$ is a diffeomorphism.)

Let $\{E_i\}$ be an orthonormal (wrt g) basis of T_pM.

This gives a diffeomorphism $E: \mathbb{R}^n \longrightarrow T_pM$

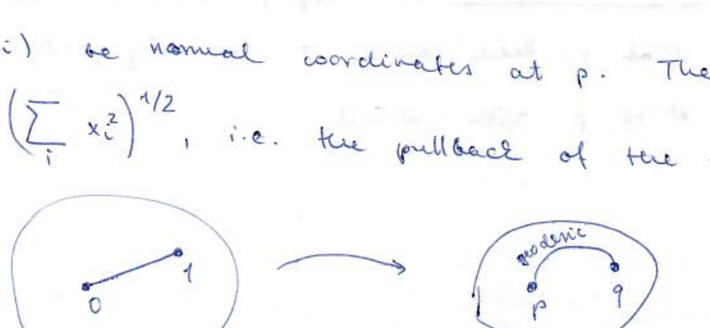
$$(x_1, \dots, x_n) \mapsto \sum_i x_i E_i$$

For a normal nbhd $U \ni p$ consider the composite

$$\varphi := E^{-1} \circ \exp_p^{-1} : U \longrightarrow \mathbb{R}^n$$

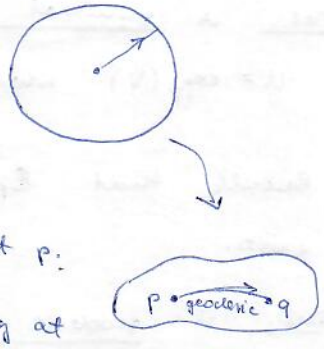
This φ is a coordinate chart called a normal coordinate chart. at p. So normal coordinate charts at p are (by construction) in 1:1 correspondence with orthonormal bases of T_pM.

Def. Let (x_i) be normal coordinates at p. The radial distance function is $r(x) = \left(\sum_i x_i^2\right)^{1/2}$, i.e. the pullback of the euclidean distance function.



Def. The unit radial vector field: $\frac{\partial}{\partial r} = \sum_i \frac{x_i}{r(x)} \frac{\partial}{\partial x_i}$

These notions are good for local computations.



Prop. (Properties of normal coordinates)

Let $(U,(x_i))$ be some normal coordinate chart centered at p:

a) For $V = \sum_i V_i \frac{\partial}{\partial x_i}\big|_p \in T_pM$ the geodesic γ_V starting at p with initial velocity V is represented in coordinates by $\gamma_V(t) = (tV_1, \dots, tV_n)$

(So this geodesic actually looks like a line segment in these coordinates.)

b) The coordinates of p are $(0,\dots,0)$.

c) The components of the metric g at p are $g_{ij} = \delta_{ij}$,

(so at p the metric looks like the euclidean metric in these coordinates).

d) Any ball $\{x \mid r(x) < \varepsilon\} \subset U$ is a geodesic ball.

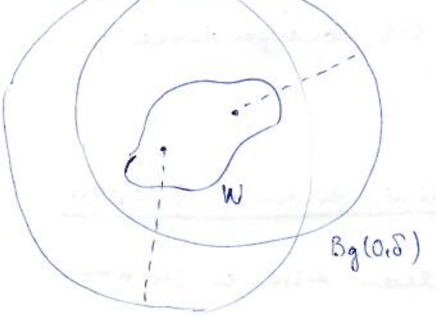
e) For any $q \in U \setminus \{p\}$: $\frac{\partial}{\partial r}\big|_q$ is the velocity vector of the geodesic from p to q and has unit g-length.

f) The first partial derivatives of g_{ij} and Γ^k_{ij} vanish at p.

Pf: follows from the definitions, exercise. □

Def. Geodesics through p and lying entirely in U are called radial geodesics. (Geodesics through other points of U do not have this simple form.)

Def. An open set $W \subset M$ is called uniformly normal if there exists a $\delta > 0$ s.t. $\forall p \in W$: $W \subset \exp_p(B_g(0,\delta))$ ("=" $B_g(0,\delta)$ by abuse of notation)



Lemma / Theorem. For any $p \in M$ and normal nbhd. U of p there exists a uniformly normal nbhd. W of p with $W \subset U$.

Global Analysis I, lecture 19

PROOF: Let $\exp_p : \mathcal{E} \longrightarrow M$ be the exp. map, and define

$$F : \mathcal{E} \longrightarrow M \times M$$
$$(q, V) \longmapsto (q, \exp_q V)$$

Let $(U, (x_i))$ be normal coordinates at p,

and let (x_i, v_i) denote the associated standard coordinates on $\pi^{-1}(U) \subset TM$.

In these coordinates the Jacobian matrix of F at $(p, 0)$ is given by

$$F_* = \begin{pmatrix} \frac{\partial x_i}{\partial x_j} & \frac{\partial x_i}{\partial v_j} \\ \frac{\partial \exp_i}{\partial x_j} & \frac{\partial \exp_i}{\partial v_j} \end{pmatrix} = \begin{pmatrix} Id & 0 \\ * & Id \end{pmatrix},$$ which is invertible.

By the inverse function theorem: F is a diffeomorphism from some nbhd $\mathcal{O}$ of $(p, 0)$ onto its image in $M \times M$.

For an open set $Y \subset M \times M$ and $\delta > 0$ define

$$Y_\delta := \{ (q, V) \in TM \mid q \in Y, \ \|V\|_g < \delta \} \subset TM,$$

which is an open set in TM

We will show that there is a set Y_δ which lies entirely in $\mathcal{O}$.

Since TM carries the product topology, there exists some $\varepsilon > 0$ s.t.

$$\mathcal{X} = \{ (x, V) \mid r(x) < 2\varepsilon, \ \|V\|_{\bar{g}} < 2\varepsilon \} \subset \mathcal{O}$$

where $\bar{g}$ denotes the euclidean metric.

Recall that there is a $C > 0$ s.t.

$$\frac{1}{C} \|V\|_g \leq \|V\|_{\bar{g}} \leq C \|V\|_g$$

holds.

Now set $Y := \{ x \in M \mid r(x) < \varepsilon \} \subset M$ and $\delta := \frac{\varepsilon}{C}$, and form the associated Y_δ. as above.

Then for $(x, V) \in Y_\delta$ it follows that $\|V\|_{\bar{g}} \leq \frac{1}{C} \|V\|_g < \varepsilon$ and hence $Y_\delta \subset X \subset \mathcal{O}$.

Now F is a diffeomorphism on Y_δ and takes $(p, 0)$ to (p, p).

Thus there is an open $W \subset M$ for which $(p,p) \in W \times W \subset F(Y_\delta) \subset M \times M$.

We may thus assume that $W \subset Y$.

Claims. (1) $\exp_q$ is a difeomorphism on $B_g(0,\delta) \subset T_qM$. $\forall q \in W$

(2) $W \subset \exp_q(B_g(0,\delta))$

Pf of (1): $\exp_q$ is defined on $B_g(0,\delta)$ and F is defined on Y_δ so $F(q,V) = (q, \exp_q V)$ is defined whenever $\|V\|_g < \delta$.

Then $F^{-1}(q,y) = (q, \varphi(q,y))$ for some $\varphi : M \times M \longrightarrow TM$ locally defined map.

Since $F^{-1} \circ F = id$ on Y_δ, we find

$$(q,V) = F^{-1} F(q,V) = F^{-1}(q, \exp_q V) = \left(q, \varphi(q, \exp_q V)\right)$$

$\Rightarrow$ $V = \varphi(q, \exp_q V)$ on $B_g(0,\delta) \subset T_qM$. $\forall q \in W$.

Similarly, doing the computation in the other direction we find $\exp_q(\varphi(q,y)) = y$.

Hence $\varphi_q(y) = \varphi(q,y)$ is a smooth inverse for $\exp_q$, proving (1).

Pf of (2): let $(q,y) \in W \times W \subset F(Y_\delta)$.

There is some $V \in B_g(0,\delta) \subset T_qM$ with $(q,y) = F(q,V) = (q, \exp_q V)$.

$\Rightarrow$ $y = \exp_q V$. $\Rightarrow$ $W \subset \exp_q(B_g(0,\delta))$, proving (2). □

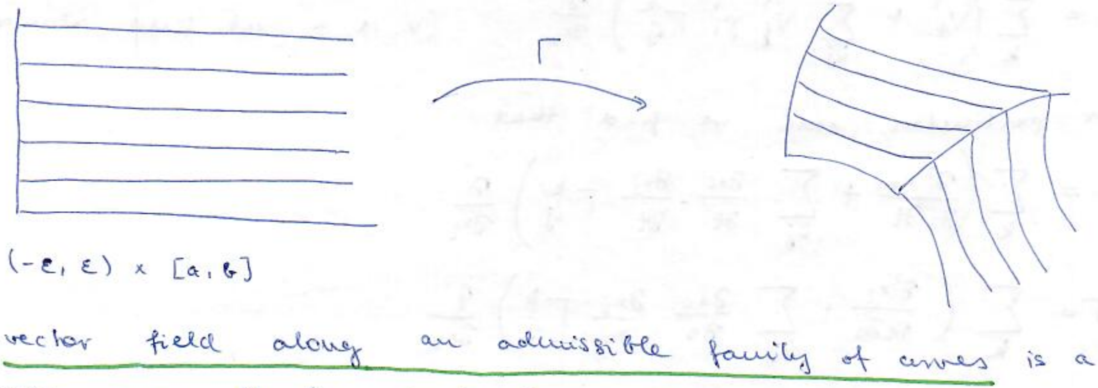
Global Analysis 1, lecture 20 9.1.2018

(M,g) is a Riemannian manifold, ∇ is the Riemannian connection

Def. A pwise smooth curve $\gamma : [a,b] \longrightarrow M$ is minimising if for any pw. smooth curve $\tilde{\gamma}$ between $p = \gamma(a)$ and $q = \gamma(b)$ we have $L(\gamma) \leq L(\tilde{\gamma})$

Observation: since $d_g(p,q) \leq L(\gamma)$ and if $d_g(p,q) < L(\gamma)$ then $\exists \tilde{\gamma} : L(\tilde{\gamma}) < L(\gamma)$ we have that if γ is minimising then $d_g(p,q) = L(\gamma)$.

Def. An admissible family of curves is a map $\Gamma : (-\varepsilon, \varepsilon) \times [a,b] \longrightarrow M$ (for some $\varepsilon > 0$) such that there is a finite subdivision $a = a_0 < a_1 < \dots < a_k = b$ for which Γ is smooth on $(-\varepsilon, \varepsilon) \times [a_{i-1}, a_i]$ and $\Gamma_s(t) := \Gamma(s,t)$ is a piecewise smooth curve for every $s \in (-\varepsilon, \varepsilon)$, which are called the main curves, and $\Gamma^t(s) := \Gamma(s,t)$ is a pw smooth curve $\forall t \in [a,b]$, these are the transverse curves.



Def. A vector field along an admissible family of curves is a continuous map $V : (-\varepsilon, \varepsilon) \times [a,b] \longrightarrow TM$ such that $V(s,t) \in T_{\Gamma(s,t)} M$ $\forall (s,t) \in (-\varepsilon, \varepsilon) \times [a,b]$ and there is a (possibly finer) subdivision $a = b_0 < \dots < b_m = b$ for which $V\big|_{(-\varepsilon,\varepsilon) \times [a,b]}$ is smooth.

Examples. $\partial_t \Gamma(s,t) := \frac{d}{dt} \Gamma_s(t)$ $\qquad$ $\partial_s \Gamma(s,t) := \frac{d}{ds} \Gamma^t(s)$

Then $\partial_s \Gamma$ is continuous on $(-\varepsilon, \varepsilon) \times [a,b]$, but $\partial_t \Gamma$ is in general not continuous at the dividing points a_i.

For a vector field V along an admissible family of curves we can calculate its covariant derivatives along Γ_s (whenever these are smooth) and along Γ^t (always).

Notation: $D_t V$: covariant derivative along Γ_s (t is the variable)

$D_s V$: covariant derivative along Γ^t (s is the (variable)

Lemma. (Symmetry lemma) Let $\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be an admissable family. Then on any rectangle $(-\varepsilon, \varepsilon) \times [a_{i-1}, a_i]$ where Γ is smooth we have $D_s \partial_t \Gamma = D_t \partial_s \Gamma$.

Pf: We only have to use the symmetry of the Riem. connection ∇.

We do the computation in local coordinates (x^i) around a point $\Gamma(s_0, t_0)$.

$\Gamma(s,t) = \left(x_1(s,t), \ldots, x_n(s,t) \right)$ in these coordinates

$$\partial_t \Gamma = \sum_k \frac{\partial x_k}{\partial t} \frac{\partial}{\partial x_k}$$

$$\partial_s \Gamma = \sum_k \frac{\partial x_k}{\partial s} \frac{\partial}{\partial x_k}$$

We use the coordinate formula for the covariant derivative:

$$D_t V = \sum_k \left(V'_k + \sum_{ij} V_j \gamma'^i \Gamma^k_{ij} \right) \frac{\partial}{\partial x_k} \qquad (V \text{ is a vct field along } \gamma)$$

In our particular case we find that

$$D_s \partial_t \Gamma = \sum_k \left(\frac{\partial^2 x_k}{\partial s \partial t} + \sum_{ij} \frac{\partial x_i}{\partial t} \frac{\partial x_j}{\partial s} \Gamma^k_{ij} \right) \frac{\partial}{\partial x_k}$$

$$D_t \partial_s \Gamma = \sum_k \left(\frac{\partial^2 x_k}{\partial t \partial s} + \sum_{i,j} \frac{\partial x_i}{\partial s} \frac{\partial x_j}{\partial t} \Gamma^k_{ij} \right) \frac{\partial}{\partial x_k}$$

By the symmetry of ∇, $\Gamma^k_{ij} = \Gamma^k_{ji}$, so by interchanging the indices i and j above we see that the formulas are identical. □

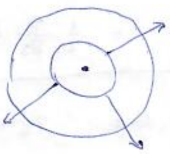
Global Analysis I, lecture 20

Thm. (Gauss Lemma) Let $\mathcal{U}$ be a geodesic ball centered at $p \in M$.

Then the unit radial vector field

$$\frac{\partial}{\partial r} = \sum_i \frac{x^i}{r(x)} \frac{\partial}{\partial x^i} \qquad \left(\text{where} \quad r(x) = \left(\sum (x^i)^2 \right)^{1/2} \right)$$

is g-orthogonal to the geodesic spheres in $\mathcal{U}$.



Pf: Let $q \in \mathcal{U}$, $X \in T_qM$ with X tangent to the geodesic sphere through q.

Let $Z \subset T_pM$ be such that $\exp_p : Z \longrightarrow \mathcal{U}$ is a diffeomorphism (i.e. take the inverse image of $\mathcal{U}$).

There is a $V \in Z \subset T_pM$ with $q \in \exp_p(V)$ and $W \in T_V(T_pM) \cong T_pM$ s.t. $X = (\exp_p)_*(W) = d(\exp_p)(W)$.

Then for $R := d(p,q)$ we have $V \in \partial B(0,R)$ and $W \in T_V(\partial B(0,R))$

$\gamma_V(t) = \exp_p(tV)$ is the radial geodesic from p to q,

and $\gamma_V'(t) = R \cdot \frac{\partial}{\partial r}\Big|_t$.

Hence $V = R \cdot \frac{\partial}{\partial r}\Big|_{t=1}$.

Then it suffices to show that $\langle X, \gamma'(1) \rangle_g = 0$.

Let $\sigma : (-\varepsilon, \varepsilon) \longrightarrow T_pM$ be any curve with $\sigma(0) = V$ and $\sigma'(0) = W$.

Consider the family $\Gamma(s,t) = \exp_p(t \cdot \sigma(s))$.

Then $\forall s \in (-\varepsilon, \varepsilon)$: $\sigma(s)$ has length R and thus Γ_s is a constant speed geodesic.

For $S := \partial_s \Gamma$ and $T := \partial_t \Gamma$ we have the following identities:

$$S(0,0) = \frac{d}{ds}\Big|_{s=0} \exp_p(0) = 0$$

$$T(0,0) = \frac{d}{dt}\Big|_{t=0} \exp_p(tV) = V$$

$$S(0,1) = \frac{d}{ds}\Big|_{s=0} \exp_p(\sigma(s)) = (\exp_p)_*(\sigma(0)) = X$$

$$T(0,1) = \frac{d}{dt}\Big|_{t=0} \exp_p(tV) = \gamma'(1)$$

Hence $\langle S, T \rangle_g = 0$ for $(s,t) = (0,0)$ and $\langle S, T \rangle_g = \langle X, \gamma'(1) \rangle$ for $(s,t) = (0,1)$.

Now show that $\langle S, T\rangle_g$ does not depend on t:

$$\frac{\partial}{\partial t}\langle S, T\rangle = \langle D_t S, T\rangle + \langle S, D_t T\rangle$$

$$= \langle D_t S, T\rangle + 0$$

$$= \frac{1}{2}\frac{\partial}{\partial s}\langle T, T\rangle = 0$$

Using
- the symmetry lemma
- $D_t T = 0$ since Γ_s is a geodesic
- $|T| = |\Gamma_s| = R \Rightarrow \langle T, T\rangle = R^2$

This gives $\langle X, \gamma'(1)\rangle = R\langle X, \frac{\partial}{\partial r}\rangle = 0$. □

Cor. Let (x^i) be normal coordinates on some geodesic ball U centered at $p \in M$. If r denotes the radial distance function, then

$$\operatorname{grad} r = \frac{\partial}{\partial r} \quad \text{on } U \setminus p.$$

Pf: Since by definition of the gradient

$$\langle \operatorname{grad} r, Y\rangle_g = dr(Y) \qquad \forall Y \in \mathfrak{X}(M)$$

we need to show that

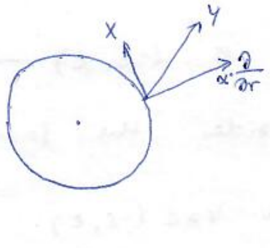
$$dr(Y) = \left\langle \frac{\partial}{\partial r}, Y\right\rangle_g$$

The geodesic sphere $\exp_p(B(0,R))$ through q is characterised by

$$r(x) = R$$

in normal coordinates. Then

$$Y = \alpha\frac{\partial}{\partial r} + X$$



for some X and α.

It holds that $dr\left(\frac{\partial}{\partial r}\right) = 1$ (via direct computation),

and $dr(X) = 0$ since X is tangent to a level set of r.

$$\rightarrow \quad dr(Y) = dr\left(\alpha\frac{\partial}{\partial r} + X\right) = \alpha$$

Now we compute $\left\langle \frac{\partial}{\partial r}, Y\right\rangle_g$ using that $\frac{\partial}{\partial r}$ is a unit vector.

$$\left\langle \frac{\partial}{\partial r}, \alpha\frac{\partial}{\partial r} + X\right\rangle_g = \alpha + \left\langle \frac{\partial}{\partial r}, X\right\rangle_g = \alpha$$

since $\left\langle \frac{\partial}{\partial r}, X\right\rangle_g = 0$ by the Gauss lemma. □

Global Analysis 1, lecture 20 - 21

<u>Prop.</u> Suppose $p \in M$ and $q \in M$ is contained in some geodesic ball around p. Then the radial geodesic from p to q is the unique minimising curve from p to q (up to reparametisation).

Pf: Let $\varepsilon > 0$ be s.t. $q \in \exp_p(B(0,\varepsilon))$ and let $\gamma : [0,R] \longrightarrow M$ be the radial geodesic from p to q. (WTS: $R = d_g(p,q)$).

Then $L(\gamma) = R$ as γ has unit speed.

We will show the following: $\forall \sigma : [a,b] \longrightarrow M$ unit speed curve we have $L(\gamma) < L(\sigma)$. This will prove the proposition.

Write $S_R = \exp_p(B(0,R))$ and define

- $a_0 \in [a,b]$: the last time when $\sigma(t) = p$;
- $b_0 \in [a,b]$: the first time after a_0 when $\sigma(t) \in S_R$.

For $t \in (a_0, b]$ write

$$\sigma'(t) = \alpha(t) \frac{\partial}{\partial r} + X(t) \quad \text{where } X(t) \text{ is tangent to the geod. sphere through } \sigma(t)$$

similarly as before but now with a time parameter.

This is an orthogonal decomposition by the Gauss lemma.

$$\Rightarrow |\sigma'(t)|^2 = \alpha(t)^2 + |X(t)|^2 \geq |\alpha(t)|^2$$

$$\alpha(t) = \left\langle \frac{\partial}{\partial r}, \sigma'(t) \right\rangle = dr(\sigma'(t))$$

Hence we have

$$L(\sigma) \geq L(\sigma|_{[a_0,b_0]}) = \lim_{\delta \to 0} \int_{a_0+\delta}^{b_0} |\sigma'(t)| \, dt \geq$$

$$\geq \lim_{\delta \to 0} \int_{a_0+\delta}^{b_0} \alpha(t) \, dt = \lim_{\delta \to 0} \int_{a_0+\delta}^{b_0} dr(\sigma(t)) \, dt =$$

$$= \lim_{\delta \to 0} \int_{a_0+\delta}^{b_0} \frac{d}{dt} r(\sigma(t)) \, dt = r(\sigma(b_0)) - r(\sigma(a_0)) =$$

$$= R = L(\gamma),$$

so now all that remains is to exclude the case when we have equality.

If $L(\sigma) = R$ then all the above inequalities are in fact equalities, hence $a_0 = a$, $b_0 = b$, and

$\int_{a_0}^{b_0} |\sigma'(t)|\, dt = \int_{a_0}^{b_0} \alpha(t)\, dt$ gives $X(t) = 0$ and so $\sigma' = \alpha \cdot \frac{\partial}{\partial r}$.

Since σ has unit speed, this yields $\alpha = 1$.

But σ and γ are both integral curves of $\frac{\partial}{\partial r}$ through $q \Rightarrow \sigma = \gamma$. □

Cor. Within a geodesic ball centered at p, the radial distance function $r(x)$ coincides with the Riemannian distance function $d_g(p,q)$:

$r(x) = d_g(p,q)$

where x are the coordinates of q.

Def. A curve $\gamma : I \to M$ is locally minimising if $\forall t_0 \in I$ has an open nbhd $\mathcal{U} \subset I$ s.t. $\gamma|\mathcal{U}$ is minimising for each pair of its points. 10.1.2018

Thm. Every Riemannian geodesic is a locally minimising curve.

Pf: $\gamma : I \longrightarrow M$ a geodesic, I open interval, $t_0 \in I$.

Take a uniformly normal nbhd W of $\gamma(t_0)$ and let $\mathcal{U} \subset I$ be the connected component of $\gamma^{-1}(W)$ containing t_0.

For $t_1, t_2 \in \mathcal{U}$ and $q_1 = \gamma(t_1)$, $q_2 = \gamma(t_2)$ the uniform normality of W implies that q_2 is contained in a geodesic ball around q_1.

(since $\exists \delta > 0 \ \forall q \in W: \ B(q,\delta) \supset W$)

Then the radial geodesic b/w q_1 and q_2 is the unique minimising curve b/w them.

Since the segment of γ b/w q_1 and q_2 is contained in this ball, γ must coincide with the radial geodesic b/w q_1 and q_2

$\Rightarrow \gamma$ is locally minimising. □

Thm. Every minimising curve is a geodesic. (up to reparametrisation)*

Pf: If $\gamma : [a,b] \longrightarrow M$ is a minimising curve, then $\forall t_0 \in [a,b]$ there is a nbhd $t_0 \in \mathcal{U} \subset [a,b]$ such that $\gamma(\mathcal{U})$ is contained in some uniformly normal nbhd W of $\gamma(t_0)$.

up to reparametrisation

Then $\forall t_1, t_2 \in \mathcal{U}$ γ coincides with the radial geodesic b/w $\gamma(t_1)$ and $\gamma(t_2)$ $\Rightarrow$ thus γ satisfies the geodesic equation around t_0.

Since t_0 was arbitrary, γ is a geodesic.

* An equivalent formulation: every constant speed minimising curve is a geodesic. (This technicality comes from the fact that while we like to think about curves as paths just physically being there, they are in fact parametrised paths.)

This last theorem completes our survey of the relationship between geodesics and shortest paths.

Def. A Riemannian manifold (M, g) is geodesically complete if every maximal geodesic is defined for all times $t \in \mathbb{R}$.

(If you start walking from a point in some direction along a geodesic, you can walk indefinitely long.)

Non-example. A ball in $\mathbb{R}^n$ is not geodesically complete. (You will leave the ball eventually.) A manifold with boundary is never geodesically complete.

Rmk. If (M, g) is geodesically complete $\Longrightarrow$. $\forall p \in M$: $\exp_p$ is defined on all of T_pM.

Thm. (Hopf – Rinow) A connected Riemannian manifold (without boundary) is geodesically complete iff it is complete as a metric space.

Pf: Suppose M is a complete metric space for the Riemannian distance d_g.

We argue by contradiction: let $\gamma : [a, b] \longrightarrow M$ be a geodesic that does not extend to $[a, b+\varepsilon)$ for every $\varepsilon > 0$.

Let $t_i \nearrow b$ be an increasing sequence, and set $q_i := \gamma(t_i)$

Wma γ is of unit speed. $\Rightarrow L(\gamma|_{[t_i, t_j]}) = |t_i - t_j|$ and

$d_g(\gamma(t_i), \gamma(t_j)) \leq |t_i - t_j| \longrightarrow 0 \Rightarrow (q_i)$ is a Cauchy sequence in M

$\Rightarrow \exists \lim q_i = q \in M$.

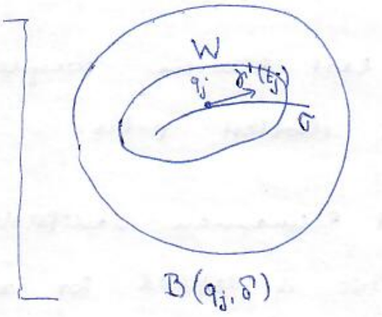
Choose a uniformly normal nbhd W of q, with $\delta > 0$: $\forall w \in W$ $W \subset B(w, \delta)$.

For j large enough, $q_j \in W$ and $t_j > b - \varepsilon$ both hold.

Since $B(q_j, \delta)$ is a geodesic ball, every geodesic starting at q_j is defined on an interval of the form $[a, a+\delta)$ (i.e. every geodesic is defined for at least time δ).

In particular, the geodesic σ starting at q_j with initial speed $\gamma'(t_j)$ is defined on $[t_j, t_j+\delta)$.

Then $\gamma\big|_{[t_j, b)} = \sigma\big|_{[t_j, b)}$ and σ extends γ past b, ↯

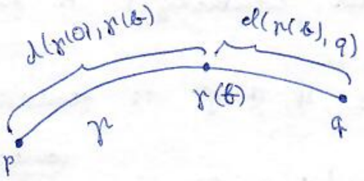


For the *other direction*, we will show that if $\exists p \in M$ for which $\exp_p$ is defined on all of T_pM, then M is complete as a metric space. (and then this property holds for $\forall p \in M$).

We first show that $\forall q \neq p$ there is a minimising geodesic segment b/w p and q.

If $\gamma : [a, b] \longrightarrow M$ is a geodesic with $\gamma(0) = p$ then we say that γ aims at q if γ is minimising

$$d(\gamma(0), q) = d(\gamma(0), \gamma(b)) + d(\gamma(b), q)$$



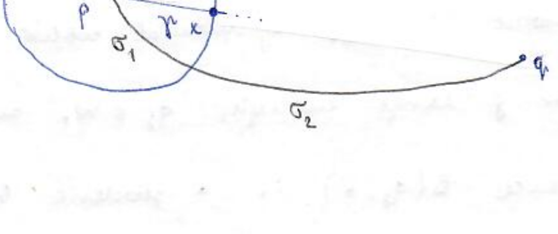
It suffices to show that there exists a geodesic starting at p, aiming at q of length $d(p,q)$, since then $\gamma(b) = q$.

Let $\varepsilon > 0$ for which $\overline{B(p, \varepsilon)}$ is a closed geodesic ball around p.

For $q \in \overline{B(p, \varepsilon)}$ we can take the radial geodesic b/w p and q and we are done, so we assume $q \notin \overline{B(p, \varepsilon)}$.

Since $m \mapsto d_g(q, m)$ is a continuous function, there exists $x \in S(p, \varepsilon)$ for which $d_g(x, q)$ is minimal on $S(p, \varepsilon)$ (since $S(p, \varepsilon)$ is compact).

Let γ be the radial geodesic from p to x. By assumption, γ is defined for all time t.



First we show that γ aims at q.

γ is already minimising, so we nts

$$d(p, q) = d(p, x) + d(x, q)$$

↯ $d(p,q) < d(p,x) + d(x,q)$ ($>$ cannot happen)

$\Rightarrow \exists$ a unit speed pw. smooth curve σ from p to q with $L(\sigma) < d(p,x) + d(x,q)$

Let σ_1 denote the part of σ inside $\overline{B(p,\varepsilon)}$ and σ_2 outside of $\overline{B(p,\varepsilon)}$

(note that σ_1 and σ_2 may not be connected paths since σ may leave and enter the ball multiple times)

$L(\sigma) \geqq \varepsilon \Rightarrow d(p,x) + d(x,q) > L(\sigma) \geqslant \varepsilon + L(\sigma_2) = d(p,x) + L(\sigma_2)$

$\Rightarrow L(\sigma_2) < d(x,q)$ contradicting the assumption that x minimises $m \mapsto d(q,m)$. ↯

So γ aims at q.

Let $T = d(p,q)$ and let $S = \{ b \in [0,T] \mid \gamma|_{[0,b]} \text{ aims at } q \}$.

Then $\varepsilon \in S$: this is what we have just shown. $\Rightarrow \sup S \geqq \varepsilon > 0$.

$A := \sup S$. WTS: $A = T$.

By continuity of the distance function, S is closed. $\Rightarrow A \in S$.

↯ Assume that $A < T$.

Let $y := \gamma(A)$ and choose $\delta > 0$ s.t. $\overline{B(y,\delta)}$ is a closed geodesic ball.

Since $A \in S$, we have $d(y,q) = d(p,q) - d(p,y) = T - A$

Choose $z \in S(y,\delta)$ that minimises $m \mapsto d(q,m)$,

and let $\tau : [0,\delta] \longrightarrow M$ be the radial geodesic from y to z.

As before, τ aims at q, so

$$d(z,q) = d(y,q) - d(y,z) = (T-A) - \delta$$

Applying the triangle inequality we find

$$d(p,z) \geqq d(p,q) - d(z,q) = T - (T - A - \delta) = A + \delta$$

Therefore the curve σ obtained by $\gamma|_{[0,A]}$ followed by τ (concatenation) is now a minimising curve from p to z.

This means that σ "has no corners", so z must lie on γ and $z = \gamma(A+\delta)$.

But then $d(p,q) = T = (A+\delta) + d(z,q) = d(p,z) + d(z,q)$

$\Rightarrow \gamma|_{[0,A+\delta]}$ aims at q and $A+\delta \in S$ ↯ $\Rightarrow A = T$.

So we have shown that there is a minimising geodesic b/w p and q.

for any p, q.

Now let q_i be a Cauchy sequence in M,

$\forall i$ let $\gamma_i(t) := \exp(t V_i)$

be a unit speed minimising geodesic from p to q.

Write $d_i = d(p, q_i) \Rightarrow q_i = \exp(d_i V_i)$.

d_i is a bounded sequence because q_i is Cauchy.

Moreover, each $V_i \in T_pM$ is a unit vector by construction.

Hence $d_i V_i$ is a bounded sequence in T_pM.

$\Rightarrow d_i V_i$ has a convergent subsequence $d_{i_k} V_{i_k}$ with limit V.

By continuity of $\exp_p$: $q_{i_k} \longrightarrow \exp_p V =: q$.

Since q_i is Cauchy, we have $q_i \longrightarrow q$. □

Curvature

16.1.2018

Def. The curvature endomorphism of a R. manifold (M, g) is the map

$$R: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

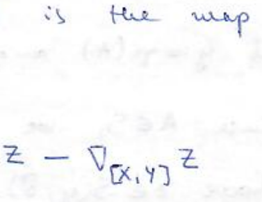
$$(X, Y, Z) \longmapsto R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

R is a (3,1)-tensor. (∇ denotes the Riemannian conn., but in the abstract sense, any connection has an associated curvature endomorphism.)

What motivates this definition?

Assume M is a surface, $p \in M$, Z_p a tangent vector.

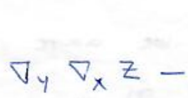
Can we extend Z_p to a vector field that is parallel? (even locally)

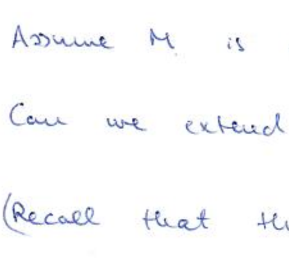


(Recall that this means $\nabla Z = 0$.)

Choose coordinates (x_1, x_2) such that p corresponds to (0,0).

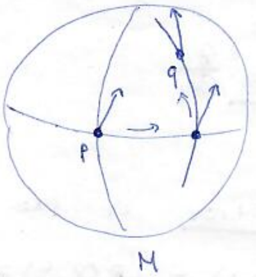
View the coordinate axes as curves through p.

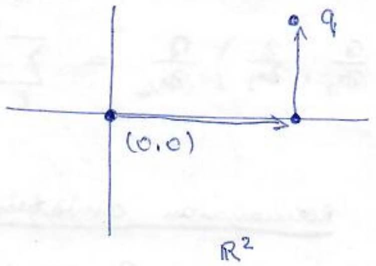




x_1-axis: γ_1 (curve)
x_2-axis: γ_2

Define a vector field Z as follows: for $q \in M$, let Z_q be obtained from Z_p by first parallel translating along γ_1 and then translate the resulting vector along γ_2.





The resulting vector field Z is parallel along γ_1 and also along every x_2-coordinate line. (by def.)

Unclear: is Z parallel along x_1-coordinate lines?

i.e. whether $\nabla_{\frac{\partial}{\partial x_1}} Z = 0$?

Since $\nabla_{\frac{\partial}{\partial x_1}} Z = 0$ whenever $x_2 = 0$, uniqueness of parallel translates gives that it suffices to show that

$$\nabla_{\frac{\partial}{\partial x_2}} \nabla_{\frac{\partial}{\partial x_1}} Z = 0.$$

This in turn would follow if we had

$$\nabla_{\frac{\partial}{\partial x_1}} \underbrace{\nabla_{\frac{\partial}{\partial x_2}} Z} = \nabla_{\frac{\partial}{\partial x_2}} \nabla_{\frac{\partial}{\partial x_1}} Z$$

since $\nabla_{\frac{\partial}{\partial x_2}} Z = 0$.

Now the relation $\nabla^{\mathbb{R}^2}_{\frac{\partial}{\partial x_1}} \nabla^{\mathbb{R}^2}_{\frac{\partial}{\partial x_2}} = \nabla^{\mathbb{R}^2}_{\frac{\partial}{\partial x_2}} \nabla^{\mathbb{R}^2}_{\frac{\partial}{\partial x_1}}$ holds for the Euclidean metric on $\mathbb{R}^2$. Note that this is a different connection than ∇.

However, for general $X, Y \in \mathfrak{X}(\mathbb{R}^2)$ we have

$$\nabla^{\mathbb{R}^2}_X \nabla^{\mathbb{R}^2}_Y - \nabla^{\mathbb{R}^2}_Y \nabla^{\mathbb{R}^2}_X = \nabla^{\mathbb{R}^2}_{[X,Y]}$$

So on the sphere, the failure of coordinate vector fields to commute locally is due to the fact that the surface is "curved".

If (M,g) is locally isometric with $\mathbb{R}^n$ (with the Euclidean metric), then $\nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]}$ and $R = 0$. (This should be intuitively clear; a rigorous proof will follow later.)

Since R is a (3,1) tensor field, we can locally write it as follows:

$$R = R_{ijk}^{\ell}\, dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x_\ell}$$

where the coefficients R_{ijk}^{ℓ} are defined by

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \sum_{\ell} R_{ijk}^{\ell}\, \frac{\partial}{\partial x_\ell}$$

Def. The Riemannian curvature tensor is the covariant (4,2) tensor field Rm given by $Rm(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle_g$.

Locally Rm is given as

$$Rm = R_{ijk\ell}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^\ell.$$

with $R_{ijk\ell} = \sum_m g_{\ell m} R_{ijk}^{m}$ where $g_{\ell m} = g\left(\frac{\partial}{\partial x_\ell}, \frac{\partial}{\partial x_m}\right)$

Lemma. The curvature tensor and endomorphism are local isometry invariants of (M,g). That is, if $\varphi : (M,g) \longrightarrow (\tilde{M}, \tilde{g})$ is a local isometry then $\varphi^*(\widetilde{Rm}) = Rm$ and $\tilde{R}(\varphi_* X, \varphi_* Y)\varphi_* Z = \varphi_*\big(R(X,Y)Z\big)$ $\forall X,Y,Z \in \mathfrak{X}(M)$.

Def. A Riemannian manifold (M,g) is flat if it is locally isometric to $(\mathbb{R}^n, g^E)$ where g^E denotes the Euclidean metric.

Thm. A Riemannian manifold is flat $\Longleftrightarrow$ $Rm \equiv 0$, i.e. the Riemannian curvature vanishes identically.

One direction (flat $\Rightarrow$ $Rm \equiv 0$) follows from the above Lemma immediately. The other direction requires some more work before we can start with the proof, in particular we need some more knowledge about general vector fields.

Def. $V, W \in \mathfrak{X}(M)$. The Lie derivative of W wrt V is the vector field

$$(\mathcal{L}_V W)_p := \frac{d}{dt}\Big|_{t=0} d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right) =$$

$$= \lim_{t \to 0} \frac{d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) - W_p}{t}$$

where $\theta_t(p) = \theta_t^V(p)$ is the flow of V around p.

Lemma. $(\mathcal{L}_V W)_p$ exists $\forall p \in M$ and it is a smooth vector field.

Pf: Let $\theta = \theta^V$ be the flow of V and $(U, (x_i))$ a smooth chart around p.

Let $\mathcal{J}_0$ be an open interval containing 0 and U_0 a open subset containing p such that $\theta : \mathcal{J}_0 \times U_0 \longrightarrow U_0$.

Then for $(t,x) \in \mathcal{J}_0 \times U_0$ write θ as $t \longmapsto (\theta_1(t,x), \dots, \theta_n(t,x))$.

For $(t,x) \in \mathcal{J}_0 \times U_0$: the derivative

$$d(\theta_{-t})_{\theta_t(x)} : T_{\theta_t(x)} M \longrightarrow T_x M$$ is given by the matrix

$$\left(\frac{\partial \theta_i}{\partial x_j}(-t, \theta(t,x)) \right)_{i,j}$$

Recall that $\theta(t,x) = \theta_t(x)$.

Hence $$\underbrace{d(\theta_{-t})_{\theta_t(x)} \left(W_{\theta_t(x)} \right)}_{(*)} = \sum_{i,j} \frac{\partial \theta_i}{\partial x_j}(-t, \theta(t,x)) \, W_j(\theta(t,x)) \frac{\partial}{\partial x_i}$$

Since θ_i and W_j are smooth, the coefficients above are smooth.

Since $\mathcal{L}_V W = \frac{d}{dt}(*)$, it follows that $\mathcal{L}_V W$ is smooth. □

We would like a better expression for $\mathcal{L}_V W$.

Thm./Lemma. Let V be a vector field on M, $p \in M$, $V_p \neq 0$. Then there exist smooth coordinates around p such that $V = \frac{\partial}{\partial x_1}$ in these coordinates.

Pf: Choose coordinates $(U, (y_i))$ around p and j such that $V_j(p) \neq 0$. (There is such a j since $V_p \neq 0$).

Let S be the hypersurface defined by $\{ y_j = 0 \}$.

Then $V_p \notin T_p S$. By shrinking the neighbourhood wma that V is nowhere tangent to S on some subset $W_0 \subset S \subset M$.

It follows that $\theta^V : (-\varepsilon, \varepsilon) \times W_0 \longrightarrow M$ is a diffeomorphism onto its image (this follows from the smoothness of flows, see Exercise 9.3.)

Write $W := \theta((-\varepsilon, \varepsilon) \times W_0) \subset M$, and let $(x_1, \dots, x_n)$ be the coordinates on $W_0 \subset S$.

Take $X : \Omega \longrightarrow W_0$ to be the inverse to these coordinates, $\Omega \subset \mathbb{R}^{n-1}$.

Then the map $\psi : (-\varepsilon, \varepsilon) \times \Omega \longrightarrow M$

$$(t, x_2, \dots, x_n) \longmapsto \theta(t, X(x_2, \dots, x_n))$$

is a diffeomorphism onto a nbhd of p.

The pushforward of ψ maps $\frac{\partial}{\partial t}$ to itself and $\theta_*\left(\frac{\partial}{\partial t}\right) = V$.

Hence $\psi_*\left(\frac{\partial}{\partial t}\right) = V$, and ψ is the required chart. □

Thm. $\mathcal{L}_V W = [V, W]$.

Pf: Set $R(V) := \{ p \in M \mid V_p \neq 0 \}$, by definition $\overline{R(V)} = \operatorname{supp} V$.

Case 1. $p \in R(V)$.

Choose coordinates $(\mathcal{U}, (x_i))$ s.t. $V = \frac{\partial}{\partial x_i}$.

Then the flow of V is $\theta_t(x) = (t, x_1, \dots, x_n)$

and $(d\theta_{-t})_{\theta_t(x)} = id$.

Thus $\forall x \in \mathcal{U}$ we have

$$(d\theta_{-t})_{\theta_t(x)} (W_{\theta_t(x)}) = (d\theta_{-t})_{\theta_t(x)} \left(\sum_j W_j(x_1 + t, x_2, \dots, x_n) \frac{\partial}{\partial x_j} \right) =$$

$$= \sum_j W_j(x_1 + t, x_2, \dots, x_n) \frac{\partial}{\partial x_j}$$

We then find

$$(\mathcal{L}_V W)_x = \frac{d}{dt}\Big|_{t=0} \sum_j W_j(x_1 + t, x_2, \dots, x_n) \frac{\partial}{\partial x_j}\Big|_x =$$

$$= \sum_j \frac{\partial W_j}{\partial x_j}(x_1, \dots, x_n) \frac{\partial}{\partial x_j}\Big|_x =$$

$$= [V, W] = \sum_{i,j} \left(V_i \frac{\partial W_j}{\partial x_i} - W_j \frac{\partial V_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} \quad \checkmark$$

Case 2. $p \in \overline{R(V)}$. In this case, the statement follows by Case 1 and continuity.

Case 3. $p \in M \setminus \operatorname{supp} V$. $\Rightarrow V = 0$ in a nbhd of p, hence

$\theta^V_t = id$ and $(d\theta_{-t})_{\theta_t(p)} W_{\theta_t(p)} = W_p$ so $\mathcal{L}_V W = 0$ and $[V, W] = 0$. □

Global Analysis 1, lecture 23 17. 1. 2018

(We work towards the flatness theorem.)

Recall: M manifold, $\theta : (-\varepsilon, \varepsilon) \times \mathcal{U} \longrightarrow M$ a flow,

$\theta(0, p) = p$, $\theta(t, \theta(s, p)) = \theta(t + s, p)$, $\theta_s \circ \theta_t = \theta_{s+t}$

If $W \in \mathfrak{X}(M)$, we say that W is invariant under θ_t if $d(\theta_t)_p(W_p) = W_{\theta_t(p)}$

Thm. For $V, W \in \mathfrak{X}(M)$ the following are equivalent:

a) V, W commute, i.e. $[V, W] = 0$.

b) W is invariant under the flow of V

c) V is invariant under the flow of W

Pf. b) ⇒ a): Write $\theta_t = \theta_t^V$ for the flow of V.

Assume$^{b)}$ $W_{\theta(t)} = d(\theta_t)_p(W_p)$. Then it follows that

$d(\theta_{-t})_{\theta_t(p)} W_{\theta(t)} = W_p$ by the chain rule. (apply $d(\theta_{-t})_{\theta_t(p}$ to both sides of the previous eq.)

Apply $[V, W]_p = (\mathcal{L}_V W)_p$, and compute $(\mathcal{L}_V W)_p$.

$$(\mathcal{L}_V W)_p = \lim_{t \to 0} \frac{1}{t}\left(d(\theta_{-t})_{\theta_t(p)} W_{\theta_t(p)} - W_p \right) = 0. \checkmark$$

c) ⇒ a): The same as b) ⇒ a).

a) ⇒ b): Assume $[V, W] = \mathcal{L}_V W = 0$.

Let $p \in M$ be arbitrary and set $X(t) := d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)})$.

Then $\underset{= \frac{d}{dt}X(t)}{\underline{X'(t)}} = d(\theta_{-t})\big(\underset{0}{\mathcal{L}_V W}\big)_{\theta_t(p)} = 0$.

$X(0) = W_p$ and thus $X(t) = W_p$ constant $\overset{\forall t}{\Rightarrow}$ $W_p = d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)})$

$\Leftrightarrow (d\theta_t)_p(W_p) = W_{\theta_t(p)}$.

a) ⇒ c): The same as a) ⇒ b). □

Rmk. $X'(t) = \frac{d}{dt} X(t) = \lim_{s \to 0} \frac{1}{s}(X(t+s) - X(t)) = \lim_{s \to 0} \frac{1}{s}\Big((d\theta_{-(t+s)})_{\theta_{t+s}(p)} W_{\theta_{t+s}(p)} -$

$- (d\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)})\Big) = \lim_{s \to 0} \frac{1}{s}\Big(d\theta_{-t}\big(d\theta_{-s} W_{\theta_{t+s}(p)} - W_{\theta_t(p)}\big)\Big) =$

$= d\theta_{-t}\Big(\lim_{s \to 0} \frac{1}{s} d\theta_{-s} W_{\theta_{t+s}(p)} - W_{\theta_t(p)}\Big) = d\theta_{-t}(\mathcal{L}_V W)_{\theta_t(p)}$

<u>Thm.</u> M an n-dimensional manifold and $(E_1, \dots, E_n)$ a local frame in a nbhd. of $p \in M$ s.t. $[E_i, E_j] = 0$ $\forall i, j$ in this nbhd.

Then there is a smooth chart $(\mathcal{U}, (x_i))$ around p s.t. $E_i = \frac{\partial}{\partial x_i}$,

i.e. our local frame is a coordinate frame.

Pf: Choose a chart $(\mathcal{U}, (s_i))$ around p such that $[E_i, E_j] = 0$ on $\mathcal{U}$.

Denote by θ_i the flow of E_i.

Then there exists some $\varepsilon > 0$ and a nbhd. Y of p s.t. for $|t_i| < \varepsilon$ we have that $\theta_{1,t_1} \circ \theta_{2,t_2} \circ \dots \circ \theta_{n,t_n} : Y \longrightarrow \mathcal{U}$.

This can be done by a finite induction argument: we can choose $\varepsilon_n > 0$ and an open $\mathcal{U}_n$ s.t. $\theta_n : (-\varepsilon_n, \varepsilon_n) \times \mathcal{U}_n \longrightarrow \mathcal{U}$,

then inductively choose $\varepsilon_i > 0$ and $\mathcal{U}_i$ s.t. $\theta_i : (-\varepsilon_i, \varepsilon_i) \times \mathcal{U}_i \longrightarrow \mathcal{U}_{i+1}$.

Then take $\varepsilon := \min_i \{\varepsilon_i\}$ and $Y := \mathcal{U}_n$.

Consider the map

$$\begin{aligned} \varphi : (-\varepsilon, \varepsilon)^n &\longrightarrow \mathcal{U} \\ (s_1, \dots, s_n) &\longmapsto \theta_{1,s_1} \circ \dots \circ \theta_{n,s_n}(0). \end{aligned}$$

Then let f be smooth.

$$(d\varphi)_s \left(\frac{\partial}{\partial s_i}\right) f = \frac{\partial}{\partial s_i}\Big|_s f(\varphi(s_1, \dots, s_n)) = \frac{\partial}{\partial s_i}\Big|_s f\left(\theta_{1,s_1} \circ \dots \circ \theta_{n,s_n}(0)\right)$$

the flows of commuting vector fields commute $\nearrow$

$$= \frac{\partial}{\partial s_i}\Big|_s f\left(\theta_{i,s_i} \circ \theta_{1,s_1} \circ \dots \circ \theta_{i-1,s_{i-1}} \circ \theta_{i+1,s_{i+1}} \circ \dots \circ \theta_{n,s_n}\right)$$

$$= E_i\big|_{\varphi(s)} f$$ since for any q the map $t \mapsto \theta_{i,t}(q)$ is an integral curve of E_i through q.

So in particular this shows that $(d\varphi)_0$ is invertible and φ is a diffeomorphism in a nbhd of 0.

Then φ^{-1} gives the derived chart. $\square$

Now we prove the flatness thm.:

(M, g) Riemannian manifold is flat $\iff$ $R_m = 0$

Pf. Flat $\Leftrightarrow$ locally isometric to $\mathbb{R}^n$ with the Euclidean metric, hence flatness $\Rightarrow$ $Rm \equiv 0$.

Suppose $Rm \equiv 0$, i.e. $\forall X, Y, Z, W \in \mathfrak{X}(M)$: $Rm(X,Y,Z,W) = \langle R(X,Y)Z, W \rangle_g = 0$, then $R(X,Y)Z = \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right)Z = 0$ by the nondegeneracy of g.

We will show that each $p \in M$ admits a parallel orthonormal frame E_i, that is, $\nabla E_i = 0$.

Fix p and choose an orthonormal basis for T_pM: $E_1|_p, \dots, E_n|_p$, and take coordinates $(\mathcal{U}, (x_i))$ s.t. $E_i|_p = \frac{\partial}{\partial x_i}\Big|_p$ (note that this holds only at p); this can be done e.g. by normal coordinates.

Furthermore assume that $\varphi(\mathcal{U}) = C_\varepsilon = \{x \in \mathbb{R}^n \mid |x_i| < \varepsilon\}$ (ε-cube) where $\varphi : \mathcal{U} \to \mathbb{R}^n$, $x \mapsto (x_1(q), \dots, x_n(q))$.

Define vector fields E_i by successive parallel translation of $E_i|_p$ along x_1, then $x_2, \dots,$ then x_n.

By the ODE theorem, smooth dependence on initial conditions gives smooth vector fields E_i.

Parallel translation preserves inner products, so E_i is an orthonormal set of vector fields, and thus it is a frame since there are n of them.

Now we will show that $\nabla_{\partial_i} E_j = 0 \ \forall (i,j)$ so as to obtain $\nabla E_j = 0$.

For fixed j we have that

$\nabla_{\partial_1} E_j = 0$ whenever $x_2 = x_3 = \dots = x_n = 0$

$\nabla_{\partial_2} E_j = 0$ whenever $x_3 = x_4 = \dots = x_n = 0$,

$\nabla_{\partial_k} E_j = 0$ whenever $x_{k+1} = \dots = x_n = 0$ in general,

(in particular we have $\nabla_{\partial_n} E_j = 0$.)

Define $M_k := \{ x \mid x_{k+1} = \dots = x_n = 0 \}$, then $\nabla_{\partial_k} E_j = 0$ on M_k.

(*) $\nabla_{\partial_1} E_j = \dots = \nabla_{\partial_k} E_j = 0$ on M_k, this we will show by induction.

Since $M_n = \mathcal{U}$, this gives $\nabla_{\partial_i} E_j$ on $\mathcal{U}$.

(*) holds for $k = 1$.

Assume (*) holds for some k.

Then $\nabla_{\partial_{k+1}} E_j = 0$ on M_{k+1} by construction, and

$\nabla_{\partial_i} E_j = 0$ for $i \leq k$, $x_{i+1} = \dots = x_n = 0$ by the inductive hypothesis,

By uniqueness of parallel translates it suffices to show that

$\nabla_{\partial_{k+1}} \nabla_{\partial_i} E_j = 0$ to conclude that $\nabla_{\partial_i} E_j = 0$, on M_{k+1} for $i \leq k$.

Since $[\partial_{k+1}, \partial_i] = 0$ and $R = 0$ we have $\nabla_{\partial_i} \nabla_{\partial_{k+1}} = \nabla_{\partial_{k+1}} \nabla_{\partial_i}$,

showing that $\nabla_{\partial_{k+1}} \nabla_{\partial_i} E_j = \nabla_{\partial_i} \nabla_{\partial_{k+1}} E_j = 0$ on M_{k+1};

this completes the induction step.

Now since the Riemannian connection is torsion free we have

$$[E_i, E_j] = \underbrace{\nabla_{E_i} E_j - \nabla_{E_j} E_i}_{\text{these vanish by the above discussion}} = 0.$$

By the previous theorem, $\forall p \in M$ there are local coordinates (x_i) with $E_i = \frac{\partial}{\partial x_i}$ in a nbhd of p, and

$$\delta_{ij} = g(E_i, E_j) = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = g^E\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

So M is locally isometric to $\mathbb{R}^n$. □

Global Analysis I, Lecture 23-24

Symmetries of the Riemann tensor

Prop: We have the identities for vector fields $X, Y, Z, W \in \mathfrak{X}(M)$:

(1) $Rm(W, X, Y, Z) = -Rm(X, W, Y, Z)$

(2) $Rm(W, X, Y, Z) = -Rm(W, X, Z, Y)$

(3) $Rm(W, X, Y, Z) = Rm(Y, Z, W, X)$

(4) $Rm(W, X, Y, Z) + Rm(X, Y, W, Z) + Rm(Y, W, X, Z) = 0$

↑ 1st Bianchi identity / algebraic Bianchi identity.

23.1.2018

Pf: (1) Follows from $R(W,X)Y = -R(X,W)Y$. We verify this directly:

$$R(W,X)Y = \nabla_W \nabla_X Y - \nabla_X \nabla_W Y - \nabla_{[X,W]} Y$$

$$= -\left(\nabla_X \nabla_W Y - \nabla_W \nabla_X Y - \nabla_{[W,X]} Y\right) = -R(X,W)Y$$

because $[W,X] = WX - XW = -(XW - WX) = -[X,W]$. ✓

(2) Supp. that $Rm(W,X,Y,Y) = 0 \quad \forall Y$, then expand

$$0 = Rm(W,X,Y+Z,Y+Z)$$

$$= \underbrace{Rm(W,X,Y,Y)}_{0} + Rm(W,X,Y,Z) + Rm(W,X,Z,Y) + \underbrace{Rm(W,X,Z,Z)}_{0}$$

So it sts $Rm(W,X,Y,Y) = 0$.

Now $W(X\langle Y,Y\rangle) = W\left(\langle \nabla_X Y, Y\rangle + \langle Y, \nabla_X Y\rangle\right)$

$$= \langle \nabla_W \nabla_X Y, Y\rangle + \langle \nabla_X Y, \nabla_W Y\rangle + \langle \nabla_W Y, \nabla_X Y\rangle + \langle Y, \nabla_W \nabla_X Y\rangle$$

and $XW(\langle Y,Y\rangle) = \langle \nabla_X \nabla_W Y, Y\rangle + \langle \nabla_W Y, \nabla_X Y\rangle + \langle \nabla_X Y, \nabla_W Y\rangle + \langle Y, \nabla_X \nabla_W Y\rangle$

By the compatibility of ∇ with the Riemannian metric one obtains

$$[W,X](\langle Y,Y\rangle) = \langle \nabla_{[W,X]} Y, Y\rangle + \langle Y, \nabla_{[W,X]} Y\rangle$$

On the other hand, by subtraction we get

$$[W,X]\langle Y,Y\rangle = (WX - XW)\langle Y,Y\rangle$$

$$= \langle \nabla_W \nabla_X Y, Y\rangle + \langle Y, \nabla_W \nabla_X Y\rangle - \langle \nabla_X \nabla_W Y, Y\rangle - \langle Y, \nabla_X \nabla_W Y\rangle$$

$$= \langle (\nabla_W \nabla_X - \nabla_X \nabla_W) Y, Y\rangle + \langle Y, (\nabla_W \nabla_Y - \nabla_Y \nabla_W) Y\rangle$$

$\Rightarrow 2\,Rm(W,X,Y,Y) = 0$

✓

(4) This will follow from the identity

$$R(W,X)Y + R(X,Y)W + R(Y,W)X = 0 \qquad \forall X,Y,W$$

We verify this identity by computing the lhs:

$$\nabla_W \nabla_X Y - \nabla_X \nabla_W Y - \nabla_{[W,X]} Y + \nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]} W +$$

$$+ \nabla_Y \nabla_W X - \nabla_W \nabla_Y X - \nabla_{[Y,W]} X =$$

$$= \nabla_W \big(\underbrace{\nabla_X Y - \nabla_Y X} \big) + \nabla_X \big(\underbrace{\nabla_Y W - \nabla_W Y} \big) + \nabla_Y \big(\underbrace{\nabla_W X - \nabla_X W} \big) -$$

$$\overset{\nabla \text{ is torsion-free}}{\downarrow} \quad - \nabla_{[W,X]} Y - \nabla_{[X,Y]} W - \nabla_{[Y,W]} X =$$

$$= \nabla_W [X,Y] + \nabla_X [Y,W] + \nabla_Y [W,X] - \nabla_{[W,X]} Y - \nabla_{[X,Y]} W - \nabla_{[Y,W]} X =$$

$$= \big[W, [X,Y] \big] + \big[X, [Y,W] \big] + \big[Y, [W,X] \big] = 0,$$

this is the Jacobi identity, which holds for general Lie brackets. However, for Lie brackets given by commutators in some associative algebra it follows simply by direct calculation.

(5) We use the identities from (4):

$$Rm(W,X,Y,Z) + Rm(X,Y,W,Z) + Rm(Y,W,X,Z) = 0$$

$$Rm(X,Y,Z,W) + Rm(Y,Z,X,W) + Rm(Z,X,Y,W) = 0$$

$$Rm(Y,Z,W,X) + Rm(Z,W,Y,X) + Rm(W,Y,Z,X) = 0$$

$$Rm(Z,W,X,Y) + Rm(W,X,Z,Y) + Rm(X,Z,W,Y) = 0$$

Sum, apply (2) to the first 2 columns ⟶ cancel out

$$\Rightarrow Rm(Y,W,X,Z) + Rm(Z,X,W,Y) + Rm(W,Y,Z,X) + Rm(X,Z,W,Y) = 0$$

By (1) and (2):

$$Rm(Y,W,X,Z) = -Rm(W,Y,X,Z) - R(W,Y,Z,X)$$

and $Rm(Z,X,Y,W) = Rm(X,Z,W,Y)$

$$\Rightarrow 2\,Rm(X,Z,W,Y) - 2\,Rm(W,Y,X,Z) = 0.$$

□

Global Analysis 1, lecture 24

Prop. (2nd Bianchi identity)

(*) $\nabla_W Rm(X,Y,Z,V) + \nabla_Z Rm(X,Y,V,W) + \nabla_V Rm(X,Y,W,Z) = 0.$

Pf: It suffices to show this for basis elements V, W, X, Y, Z of some (fixed) frame, then we are done by linearity considerations. So choose normal coordinates (x_i) around a point p and choose V, W, X, Y, Z amongst $\frac{\partial}{\partial x_i}$.

Then we have $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ and also $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\Big|_p = 0$ since $\Gamma_{ij}^k(p) = 0$. Therefore at p the following holds:

$$\nabla_W Rm(Z,V,X,Y) = \nabla_W \langle R(Z,V)X, Y\rangle$$
$$= \langle \nabla_W \nabla_Z \nabla_V X - \nabla_W \nabla_V \nabla_Z X, Y\rangle$$

Permute W, Z, V cyclically and add:

$$\nabla_W Rm(Z,V,X,Y) + \nabla_Z Rm(V,W,X,Y) + \nabla_V Rm(W,Z,X,Y) =$$
$$= \langle \nabla_W \nabla_Z \nabla_V X - \nabla_W \nabla_V \nabla_Z X + \nabla_Z \nabla_V \nabla_W X - \nabla_Z \nabla_W \nabla_V X +$$
$$+ \nabla_V \nabla_W \nabla_Z X - \nabla_V \nabla_Z \nabla_W X, Y\rangle = \dagger$$

because $\nabla_V X = \nabla_W X = \nabla_Z X = 0.$ at p. and R is a $C^\infty(M)$-linear endom. Here we use that (*) is $C^\infty(M)$-linear (i.e. a tensor) we use the previous Prop. and the fact that the nonlinear terms all carry a df (which will cancel out)

$$\dagger = \langle R(W,Z)\nabla_V X + R(V,W)\nabla_Z X + R(Z,V)\nabla_W X, Y\rangle. = 0.$$

□

Ricci and scalar curvatures

Def. The Ricci tensor is the covariant 2-tensor field Rc:

$$(X,Y) \longmapsto \mathrm{Tr}_g \left(Z \longmapsto R(Z,Y)X \right)$$

In coordinates: $Rc = \sum_{i,j} R_{ij}\, dx_i \otimes dx_j = \sum_{i,j} \sum_{k,m} g^{km} R_{kijm}\, dx_i \otimes dx_j$

Def. The scalar curvature S is the fuction

$$S = \mathrm{Tr}_g (Rc) = \sum_{i,j} g^{ij} R_{ij} .$$

Prop. / Lemma. (M,g) a 2-dimensional Riemannian manifold. Let $p \in M$ and E_1, E_2 be a basis for T_pM. Then:

$$Rm(X,Y,Z,W)_p = \frac{1}{2} S_p \left(\langle X,W \rangle \langle Y,Z \rangle - \langle X,Z \rangle \langle Y,W \rangle \right)$$

$$Rc(X,Y)_p = \frac{1}{2} S_p \langle X,Y \rangle$$

$$S_p = 2 \cdot \frac{Rm(E_1,E_2,E_2,E_1)}{|E_1|^2 |E_2|^2 - \langle E_1,E_2 \rangle^2}$$

First assume that E_1, E_2 is an orthonormal basis of T_pM, and write the components of Rm in this basis.

$$R_{ijkl} = Rm(E_i, E_j, E_k, E_l) \qquad \forall\, i,j,k,l \in \{1,2\}$$

$$\Rightarrow \frac{Rm(E_1,E_2,E_2,E_1)}{|E_1|^2 |E_2|^2 - \langle E_1,E_2 \rangle^2} = R_{1221} \quad \text{by orthonormality}$$

By previous properties: $R_{1221} = R_{2112} = -R_{1212} = -R_{2121}$

Now $Rm(X,Y,Z,W) = \frac{1}{2} S \left(\langle X,W \rangle \langle Y,Z \rangle - \langle X,Z \rangle \langle Y,W \rangle \right)$ follows by inspection (simple linear algebra)

The components of Rc in this basis are:

$$R_{ij} = R_{1ij1} + R_{2ij2}$$

$$\Rightarrow R_{12} = R_{21} = 0 \quad \text{and} \quad R_{11} = R_{22} = \frac{1}{2} S_p$$

Hence $R(X,Y) = \sum_{i,j} X_i Y_j R_{i,j} = \frac{1}{2} S_p (X_1 Y_1 + X_2 Y_2)$

$$= \frac{1}{2} S_p (\langle X, Y \rangle)$$

$\Rightarrow \quad S = \mathrm{Tr}_g (Rc) = R_{11} + R_{22} = 2\,\mathrm{Rm}(E_1, E_2, E_2, E_1)$

If E_1, E_2 are not orthonormal $\longrightarrow$ replace them by

$F_1 := \frac{E_1}{|E_1|}$ and $F_2 := \dfrac{E_2 - \langle E_2, E_1 \rangle E_1}{\left| E_2 - \frac{\langle E_2, E_1 \rangle E_1}{|E_1|^2} \right|}$

i.e. Gram − Schmidt orthogonalise, and then we are done. ☐

Gauß − Bonnet theorem

24.1.2018

(M,g) compact oriented Riemannian 2-manifold

Thm. $\int_M S \, dV_g = 4\pi \chi(M)$.

Let $\gamma : [a,b] \longrightarrow \mathbb{R}^2$ be a piecewise smooth curve with unit speed

If γ is actually smooth, define the tangent angle $\vartheta : [a,b] \longrightarrow \mathbb{R}$ to be the unique map for which $\gamma'(t) = (\cos(\vartheta(t)), \sin(\vartheta(t)))$ and $\vartheta(a) \in (-\pi, \pi]$.

The smoothness of ϑ follows from the diagram:

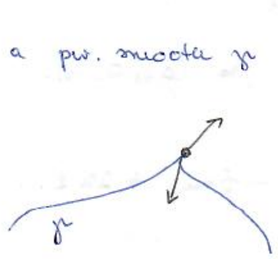
$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\pi"] \\ \gamma' : [a,b] \arrow[ur, dashed, "\vartheta"] \arrow[r] & S^1 \end{tikzcd}$$

π universal covering, local diffeomorphism

γ' is smooth, π is a local diffeom. $\Rightarrow \vartheta$ is smooth.

Def. Suppose $\gamma(a) = \gamma(b)$, $\gamma'(a) = \gamma'(b)$ and γ is smooth. (i.e. there are no "corners"). Then the rotation angle of γ is defined to be $\vartheta(b) - \vartheta(a) =: \mathrm{Rot}(\gamma) \in 2\pi\mathbb{Z}$

For a pw. smooth γ we need to deal with "jumps".



$\gamma'(c^+)$ is the right hand tangent vector at $\gamma(c)$,

$\gamma'(c^-)$ is the left hand tangent vector at $\gamma(c)$, $c \in (a,b)$.

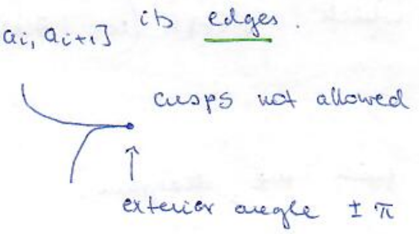
Def. Write ε_i for the exterior angle between $\gamma(a_i^-)$ and $\gamma(a_i^+)$ where $a = a_0 < a_1 < \dots < a_k = b$ is the subdivision of $[a,b]$ for which $\gamma|_{[a_i,a_{i+1}]}$ is smooth; this is defined to be the oriented angle between $\gamma'(a_i^-)$ and $\gamma'(a_i^+)$. ε_i is chosen in $[-\pi,\pi]$ and is given positive sign if $(\gamma'(a_i^-), \gamma'(a_i^+))$ is an ordered basis and negative sign otherwise.

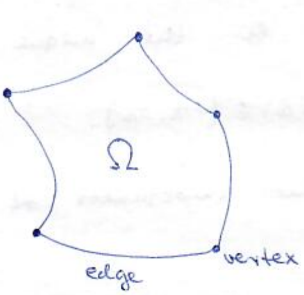
If $\gamma'(a_i^-) = -\gamma'(a_i^+)$, the exterior angle is not defined. (no canonical choice b/w $-\pi$ and $+\pi$).

no self-intersections

Def. A curved polygon in $\mathbb{R}^2$ is a simple closed piecewise smooth unit speed curve without exterior angles of $\pm\pi$ that is the boundary of an open set $\Omega \subset \mathbb{R}^2$.*

The points $\gamma(a_i)$ are called the vertices of the curved polygon and the paths $\gamma|_{[a_i,a_{i+1}]}$ its edges.





*This is necessary to exclude weird space-filling curves.

Def. γ is positively oriented if at its smooth points γ' is compatible with the Stokes orientation on $\partial\Omega$.

For such a curve γ we define the tangent angle $\vartheta : [a,b] \to \mathbb{R}$ as follows: choose $\vartheta(a) \in (-\pi,\pi]$ and for $t \in (a,a_1)$ define $\vartheta(t)$ as before. At a_1 we set $\vartheta(a_1) := \lim_{t \nearrow a_1} \vartheta(t) + \varepsilon_1$. Proceed inductively: define ϑ on (a_{i-1}, a_i) and let $\vartheta(a_i) := \lim_{t \nearrow a_i} \vartheta(t) + \varepsilon_i$, then extend to (a_i, a_{i+1}).

This gives $\vartheta : [a,b] \longrightarrow \mathbb{R}$, and define $\mathrm{Rot}(\gamma) := \vartheta(b) - \vartheta(a) \in 2\pi\mathbb{Z}$.

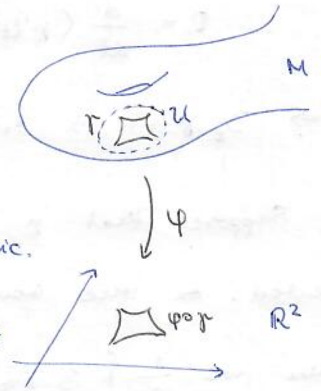
Global Analysis I, lecture 25

Thm. (Hopf) If γ is a positively oriented curved polygon in $\mathbb{R}^2$ then

$$\mathrm{Rot}(\gamma) = 2\pi.$$

Pf, see Lee's book. The proof is elementary. □

Def. (M,g) Riemannian 2-manifold. A curved polygon in M is a pw. smooth curve $\gamma : [a,b] \longrightarrow M$ that is a bdry of an open set $\Omega \subset M$ with compact closure s.t. γ is contained in a single chart $(\mathcal{U},\varphi)$ for which $\varphi \circ \gamma$ is a curved polygon in $\mathbb{R}^2$.



We can thus think of curved polygons in M as a curved polygon in some open set in $\mathbb{R}^2$ but with metric g not necessarily the Euclidean metric.

Define the exterior angles ε_i using the g-inner product (and we require the Stokes orientation) so

$$\cos \varepsilon_i = \left\langle \gamma'(a_i^+), \gamma'(a_i^-) \right\rangle_g$$

We define the tangent angles as well, this puts particular significance on $\frac{\partial}{\partial x}$, thus it is unclear whether the tangent angle ϑ obtained this way is independent of the choice of coordinates $(\mathcal{U},\varphi)$.

"We can declare these worries fake news."

We still define $\mathrm{Rot}_g(\gamma) := \vartheta(b) - \vartheta(a)$.

Lemma. γ is a positively oriented curved polygon in (M,g) $\Rightarrow$ $\mathrm{Rot}(\gamma) = 2\pi$.

Pf: Set $g_s := s\cdot g + (1-s)\,\overline{g}$ where $\overline{g}$ denotes the Eu. metric on $\mathcal{U} \subset \mathbb{R}^2$ and $s \in [0,1]$. These form a family of Riemannian metrics.

Set $f(s) := \frac{1}{2\pi} \mathrm{Rot}_{g_s}(\gamma)$.

The function f is continuous and integer-valued $\to$ constant.

And $f(0) = \frac{1}{2\pi} \mathrm{Rot}_{\overline{g}}(\gamma) = \frac{2\pi}{2\pi}$ by Hopf's thm. □

Let $N(t)$ be the normal vector field along γ that makes $(\gamma'(t), N(t))$ into an oriented basis.

Then $N(t)$ is the inward pointing normal (because we are using the Stokes orientation).

The signed curvature at a smooth point $\gamma(t)$ is defined to be

$$\varkappa_N(t) := \langle D_t(\gamma'(t)), N(t)\rangle_g$$

Since $1 = |\gamma'(t)|^2 = \langle \gamma'(t), \gamma'(t)\rangle_g$ we see that

$$0 = \frac{d}{dt}\langle \gamma'(t), \gamma'(t)\rangle = 2\langle D_t\gamma'(t), \gamma'(t)\rangle$$

$\Rightarrow$ $D_t\gamma'(t)$ is orthogonal to $\gamma'(t)$ $\Rightarrow$ $D_t\gamma'(t) = \varkappa_N(t)N(t)$.

Thm. Suppose that γ is a curved polygon in (M,g) and γ is positively oriented. on the boundary of some open set Ω.

Then
$$\frac{1}{2}\int_\Omega S\,dV_g + \int_\gamma \varkappa_N\,dS + \sum_{i=1}^{k}\varepsilon_i = 2\pi$$

Pf: Let $a = a_0 < a_1 < \ldots < a_{k-1} < a_k = b$ be the subdivision for which $\gamma|_{[a_{i-1}, a_i]}$ is smooth (i.e. the edge subdivision). We have

$$2\pi = \sum_{i=1}^{k}\varepsilon_i + \sum_{i=1}^{k}\int_{a_{i-1}}^{a_i}\vartheta'(t)\,dt = \sum_{i=1}^{k}\varepsilon_i + \sum_{i=1}^{k}(\vartheta(a_i) - \vartheta(a_{i-1})) = 0.$$

Let (x,y) be oriented coordinates on an open set $\mathcal{U}$ containing $\overline{\Omega}$ and let $(X,Y) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ be the coordinate frame.

Then $E_1 := \frac{X}{|X|}$, $E_2 := \dfrac{Y - \langle X,Y\rangle\cdot\frac{X}{|X|^2}}{\left|Y - \langle X,Y\rangle\cdot\frac{X}{|X|^2}\right|}$ is an orthonormal frame.

By construction, E_1 is a positive multiple of $\frac{\partial}{\partial x}$.

Now $\vartheta(t)$ represents the g-angle b/w $\frac{\partial}{\partial x}$ and $\gamma'(t)$ (by construction).

Hence $\gamma'(t) = \cos(\vartheta(t))E_1 + \sin(\vartheta(t))E_2$,

$$N(t) = -\sin(\vartheta(t))E_1 + \cos(\vartheta(t))E_2.$$

Now $D_t \gamma' = (-\vartheta' \sin\vartheta\, E_1 + \cos\vartheta\, (\nabla_{\gamma'} E_1) + \vartheta' \cos\vartheta\, E_2 + \sin\vartheta\, (\nabla_{\gamma'} E_2)$

$$= \vartheta'(t)\, N(t) + \cos\vartheta(t) \cdot \nabla_{\gamma'} E_1 + \sin\vartheta(t) \cdot \nabla_{\gamma'} E_2$$

Since (E_1, E_2) is orthonormal, we have $\langle E_i, E_i\rangle = 1$, $\langle E_i, E_j\rangle = 0$ $(i \neq j)$

So $0 = \nabla_X \langle E_i, E_i\rangle = 2 \langle \nabla_X E_i, E_i\rangle$

$$0 = \nabla_X \langle E_i, E_j\rangle = \langle \nabla_X E_i, E_j\rangle + \langle E_i, \nabla_X E_j\rangle$$

Now define a form ω by $\omega(X) = \langle E_1, \nabla_X E_2\rangle = \langle \nabla_X E_1, E_2\rangle$

$\Rightarrow \nabla_X E_1 = -\omega(X) E_2$, $\nabla_X E_2 = \omega(X) E_1$

$\nabla_X E_1$ is orthogonal to E_1, hence a multiple of E_2, same for 2.

(In fact, $\omega = \omega_{21} = -\omega_{12}$ connection 1-form wrt. (E_1, E_2) and $\omega_{11} = \omega_{22} = 0$.)

Now $\varkappa_N = \langle D_t \gamma', N\rangle = \langle \vartheta' N, N\rangle + \cos\vartheta \langle \nabla_{\gamma'} E_1, N\rangle + \sin\vartheta \langle \nabla_{\gamma'} E_2, N\rangle$.

$$= \vartheta' - \cos\vartheta \langle \omega(\gamma') E_2, N\rangle + \sin\vartheta \langle \omega(\gamma') E_1, N\rangle$$

$$= \vartheta' - \cos^2\vartheta\, \omega(\gamma') - \sin^2\vartheta\, \omega(\gamma')$$

$$= \vartheta' - \omega(\gamma')$$

$\Rightarrow \vartheta' = \varkappa_N + \omega(\gamma')$.

Then we find $2\pi = \sum_{i=1}^{k} \varepsilon_i + \sum_{i=1}^{k} \int_{a_{i-1}}^{a_i} \varkappa_N(t)\, dt + \underbrace{\sum_{i=1}^{k} \int_{a_{i-1}}^{a_i} \omega(\gamma')\, dt}$

Remains to show: $= \int S\, dV_g$

Approximate Ω by open sets Ω_j with smooth bdry γ_j so that

$L(\gamma) = \lim_{j\to+\infty} L(\gamma_j)$ and $\text{area}(\Omega) = \lim_{j\to+\infty} \text{area}(\Omega_j)$

$$\Rightarrow \int_\gamma \omega = \lim_j \int_{\gamma_j} \omega = \lim_j \int_{\Omega_j} d\omega = \int_\Omega d\omega$$

Need to compute $d\omega$; it suffices to compute $d\omega(E_1, E_2) = -d\omega(E_2, E_1)$

$d\omega(E_1, E_2) = E_1(\omega(E_2)) - E_2(\omega(E_1)) - \omega([E_1, E_2])$

scalar multiplication

action

$= \langle E_1 \cdot (E_1(\omega(E_2))) + \omega(E_2) \nabla_{E_1} E_1 - E_1 \cdot (E_2(\omega(E_1))) -$

use that $\langle \nabla_X E_1, E_1\rangle = 0$

$- \omega(E_1) \nabla_{E_2} E_1 - \omega([E_1, E_2]) E_1, E_1\rangle =$

[illegible] does not alter $\langle \cdot, \cdot \rangle$

$$= \left\langle \nabla_{E_1}(E_1 \cdot \omega(E_2)) - \nabla_{E_2}(E_1 \cdot \omega(E_1)) - \omega([E_1, E_2]) E_1, \ E_1 \right\rangle =$$

verify using the Leibniz rule for ∇

$$= Rm(E_1, E_2, E_2, E_1) = \frac{1}{2} S$$

□

Note: this thm says that a curved polygon is what we think it is. 30.1.2018

$\varkappa_N(t) = \langle D_t \gamma'(t), N(t) \rangle$ at smooth points

$D_t \gamma'(t) = \varkappa_N(t) N(t)$

$N(t) \perp \gamma$ with Stokes orientation

Cor. 1. The sum of the angles of a Euclidean triangle is π.

Cor. 2. The circumference of a Euclidean circle of radius R is $2\pi R$.

Cor. 3. If γ is a smooth simple closed curve in $\mathbb{R}^2$ of unit speed then

$$\int_a^b \varkappa_N(t)\, dt = 2\pi.$$

Thm. (Gauß–Bonnet for surfaces) If (M, g) is compact, oriented, 2-dimensional

$$\Rightarrow \int_M S \, dV_g = 4\pi \chi(M).$$

Def. Let (M, g) be a smooth compact surface. A triangulation of M is a finite collection $\mathcal{T} = \{T_i\}_{i \in I}$ of curved triangles $T_i \subset M$ (i.e. curved polygons with 3 vertices) with the following properties:

- $T_i = \partial \overline{\Omega_i}$ where Ω_i is an open set with compact closure
- $\bigcup_i \overline{\Omega_i} = M$
- $\forall i \neq j: \quad T_i \cap T_j = \begin{cases} \emptyset \\ \text{single vertex} \\ \text{single edge} \end{cases}$

"I'm gonna do sth terrible and write words on the right side of this eq."

Prop. (PO) Every smooth compact surface admits a triangulation. □

Given a triangulation $\mathcal{T}$ of M define

$N_v := \#$ vertices $= \# \bigcup_i \{v \text{ is a vertex of } T_i\}$ number of vertices (w/o multiplicity)

$N_e := \#$ edges $= \# \bigcup_i \{e \text{ is an edge of } T_i\}$ number of edges (w/o multiplicity)

$N_f := \#$ faces $= \sum_i \# \{\Omega_i \mid T_i = \partial \overline{\Omega_i}\} = \# \{T_i\}$ number of faces

Global Analysis I, lecture 26

Def. Euler characteristic: $\chi(M,T) := N_v - N_e + N_f$

Prop. $\chi(M,T)$ does not depend on T.

Pf: Subdivide, elementary graph theory. □

⇒ $\chi(M)$ is an invariant of M.

Thm: (M,g) triangulated compact oriented surface ⇒ $\int_M S\, dV_g = 4\pi\chi(M)$.

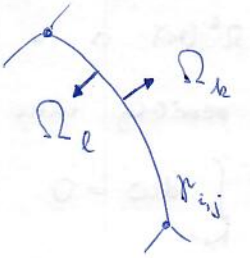
Pf: Fix a triangulation T. Let $\{\Omega_i\}_{i\in I}$ be the set of faces for T, $\{\gamma_{i,j} \mid j=1,2,3\}$ edges of T_i $(i\in I)$, $\{\vartheta_{i,j} \mid j=1,2,3\}$ interior angles of T_i (or Ω_i, $i\in I$), ⇒ $\varepsilon_{i,j} = \pi - \vartheta_{i,j}$ exterior angles.

$$\sum_{i=1}^{N_f}\left(\int_{\Omega_i}\frac{1}{2} S\, dV_g + \sum_{j=1}^{3}\int_{\gamma_{ij}} \varkappa_N\, dS + \sum_{j=1}^{3}\left(\pi - \vartheta_{i,j}\right)\right) \underset{\uparrow}{=} \sum_{i=1}^{N_f} 2\pi$$

by the prev. Thm.

$\gamma_{i,j}$ bound the region Ω_i

⇒ each of the $\int_{\gamma_{i,j}} \varkappa_N$ appears twice, but with opposite orientations

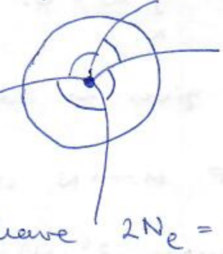


⇒ $\sum_i \sum_j \int_{\gamma_{ij}} \varkappa_N = 0$

So we get:

$$\frac{1}{2}\int_M S\, dV_g + 3\pi N_f - \underbrace{\sum_{i=1}^{N_f}\sum_{j=1}^{3} \vartheta_{i,j}}_{2\pi N_v} = 2\pi N_f$$

At each vertex, the angles touching that vertex sum up to 2π.



⇒ $\frac{1}{2}\int_M S\, dV_g = 2\pi N_v - \pi N_f$

Since each edge occurs at 2 triangles exactly, we have $2N_e = 3N_f$

⇒ $2N_e - 2N_f = N_f$

$$\Rightarrow \frac{1}{2}\int_M S dV_g = 2\pi N_v - \pi N_f =$$
$$= 2\pi N_v - 2\pi N_e + 2\pi N_f$$

□

Let M be a manifold.

Recall the notion of the exterior derivative $d: \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ on forms.

$$\underbrace{d^2=0}_{d_{k+1}\circ d_k=0}, \qquad \Omega^k \xrightarrow{d_k} \Omega^{k+1} \xrightarrow{d_{k+1}} \Omega^{k+2} \longrightarrow \dots$$

Define $Z^k(M) := \{\omega \in \Omega^k(M) \mid d\omega = 0\} = \ker d_k$ closed forms

$B^k(M) := \{\omega \in \Omega^k(M) \mid \exists \eta \in \Omega^{k-1}(M): d_{k-1}\eta = \omega\} = \operatorname{im} d_{k-1}$ exact forms

By $d_k \circ d_{k-1} = 0$ we have $B^k(M) \subset Z^k(M)$

The k^{th} de Rham cohomology group of M is the $\mathbb{R}$-vector space

$$H^k(M) = Z^k(M) \big/ B^k(M)$$

This will coincide with the singular cohomology of the smooth manifold; the point is that since we have this nice differential structure, computations are easier than if we were to treat M simply as a topological space.

Suppose $\omega \in \Omega^k(M)$ is a closed form, $N \subset M$ is an oriented embedded k-dimensional submanifold possibly with boundary ∂N. Then

$$\int_{\partial N} \omega = \int_N d\omega = 0 \quad \text{by Stokes}$$

Similarly, if ω is exact then $\int_N \omega = \int_N d\eta = \int_{\partial N} \eta$, hence if $\partial N = \emptyset$ then $\int_N \omega = 0$.

This gives us criteria for checking non-closedness and non-exactness.

If $F: M \longrightarrow N$ is a smooth map, we have $F^*: \Omega^k(N) \longrightarrow \Omega^k(M)$ pullback, F^* maps $Z^k(N) \to Z^k(M)$ and $B^k(N) \to B^k(M)$. (This follows from the fact that the pullback and the differential commute)

$\Rightarrow$ induced map $F^*: H^k(N) \longrightarrow H^k(M)$, in particular for $id: M \longrightarrow M$ we get $(id)^* = id$ on each $H^k(M)$, and $F^* \circ G^* = (G \circ F)^*$.

Hence if M and N are difeomorphic then $H^k(M) \cong H^k(N)$ $\forall k$.

Global Analysis I, lecture 26

Recall: $F_0, F_1 : M \longrightarrow N$ are homotopic if $\exists\, F : M \times [0,1] \to N$ continuous s.t. $F(-,0) = F_0$, $F(-,1) = F_1$.

Under reasonable conditions, a continuous homotopy can be made smooth.

Thm. $F_0, F_1 : M \longrightarrow N$ are homotopic $\Rightarrow$ $F_0^* = F_1^*$ induced maps in cohomology. □

Cor. M, N htp. equivalent $\Rightarrow$ $H^k(M) \cong H^k(N)$. □

Ex. $H^k(\mathbb{R}^n) \cong H^k(\mathbb{H}^n) \cong H^k(*)$

Ex. $H^*(S^n) \cong \begin{cases} \mathbb{R} & * = 0 \text{ or } * = n \\ 0 & 0 < * < n \end{cases}$

Any choice of orientation form $\omega \in \Omega^n(S^n)$ generates $H^n(S^n)$

Ex. $n \geq 2$, $x \in \mathbb{R}^n$, $M := \mathbb{R}^n \setminus \{x\}$ $\Rightarrow$ $M \simeq S^{n-1}$ $\Rightarrow$ $H^0(M) \cong H^{n-1}(M) = \mathbb{R}$, all other are 0.

Cor. $\mathcal{U} \subseteq \mathbb{R}^n$ open $\Rightarrow$ $H^n(\mathcal{U} \setminus \{x\}) \not\cong 0$. (This can be seen using the Mayer-Vietoris sequence.)

$\Rightarrow$ manifolds of different dimension cannot be homeomorphic.

Thm. M cpt. $\Rightarrow$ $H^k(M)$ is finite dimensional $\forall k > 0$. □

Def. For a cpt. manifold M define the Euler characteristic as

$$\chi(M) := \sum_{k=0}^{\dim M} (-1)^k \cdot \dim\left(H^k(M)\right).$$

Thm. (Gauß-Bonnet-Chern) (M, g) cpt. oriented Riemannian n-manifold. Then

$$\chi(M) = \frac{1}{(2\pi)^n} \int_M \mathrm{Pf}(-R)\, dV_g$$ where R is the curvature endomorphism

and Pf denotes the Pfafian.

Global Analysis I, lecture 27 31. 1. 2018

Exam starts at 9:00 s.t., arrive at 8:50.

GHS, Wegelerstr. 10

Klausureinsicht: will be announced on the website: one b/w the two exams and after the second. Results can be expected by 05.02.

Problems like problem sheets:

+ be able to use the theory & results discussed in the lecture
+ give basic definitions of the notions
- theorems won't have to be meticulously formulated
- proofs of big results won't be asked
- 1st exam: no questions about this or last week (this basically means Gauß – Bonnet)

Content

Partitions of unity

Tangent vectors as derivations, tangent bundle of a manifold

Differential of a smooth map

Vector bundles (in general), local trivialisations, sections (e.g. vector fields: TM)

Cotangent bundle, T^*M and its sections $\Omega^1(M)$

Alternating tensors (bundles thereof)

Differential k-forms as sections of bundles of alternating tensors

Wedge product of forms $\Omega^k \times \Omega^l \to \Omega^{k+l}$, pullback of forms

Orientation, orientation forms

For manifolds with boundary: boundary orientation, normal vector field

Integration on manifolds (with or without boundary), Stokes' theorem

(no Riem. structure needed)

Riemannian metric $g: \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$.

Tangent-cotangent isomorphism $\mathfrak{X}(M) \xrightarrow{\sim} \Omega^1(M)$

Orientation + metric → volume form → integral

Divergence, divergence thm. (Riem. structure needed)

Gradient (Riem.), boundary defining function (for a manifold with boundary)

Piecewise smooth curves, line integral, length of a curve → Riem. distance function, the metric topology coincides with the manifold topology

Tensor bundles $T^k_\ell M = (T^*M)^{\otimes k} \otimes (TM)^{\otimes \ell}$, tensor fields as sections

Connections: $\mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ (linear) Leibniz map: $\nabla_X (Y \cdot f) = (\nabla_X Y) f + Y(X \cdot f)$

Extension of lin. connections to various tensor bundles

Vector fields along a curve, covariant derivative along a curve

Geodesics, parallel vector fields

Existence and uniqueness of geodesics with given initial velocity vector and initial point

- Riemannian connection: compatible with g and torsion-free. Existence & uniqueness.

Existence and uniqueness of flows, smooth dependence on initial conditions

(the details of the proof definitely won't be asked for)

Exponential map $\mathcal{E} \longrightarrow M$

Normal coordinates, uniformly normal coordinates

Minimising properties

Geodesic completeness vs. metric completeness, Hopf – Rinow

Curvature, curvature tensor (Riemann tensor)

Lie derivative, $\mathcal{L}_V W = [V, W]$

Flatness criterion for Riemannian manifolds

Gauß – Bonnet thm.